# THE SEVEN MOST IMPORTANT EQUATIONS FOR YOUR RETIREMENT:

## From Conversations to Calculations

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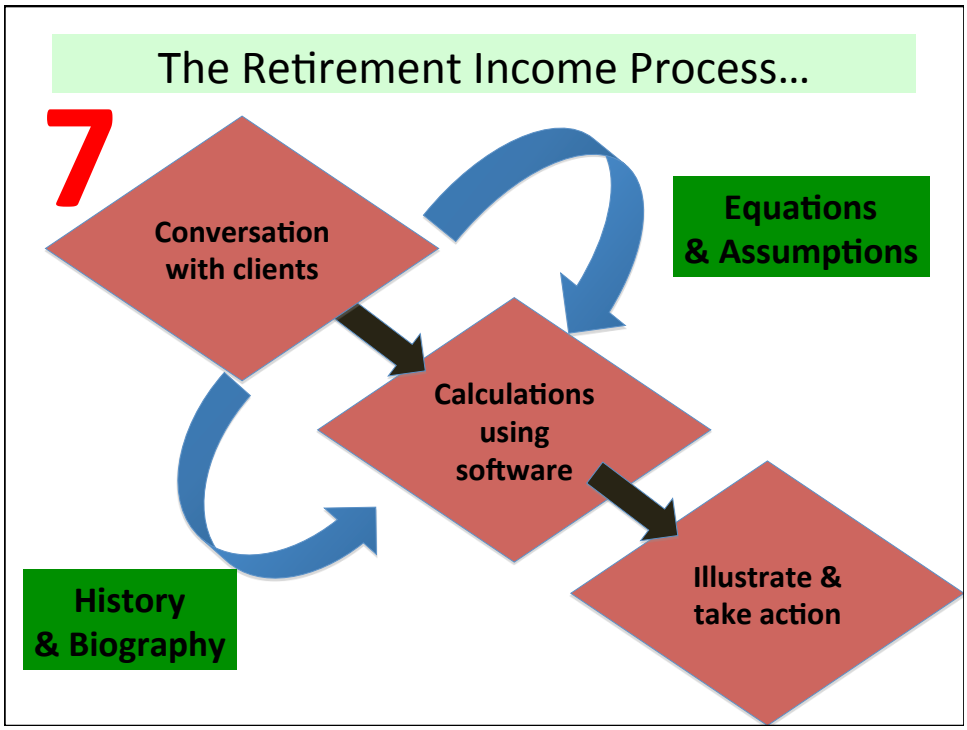
**The IFID Centre**
**Toronto**

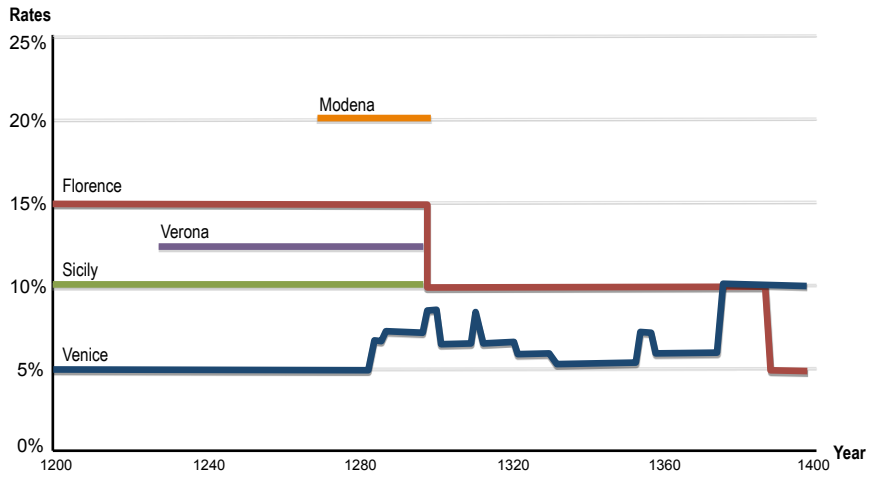


**Conversation #1:**
**If you stop working today, invest your nest egg as safely as possibly, and try to maintain your current standard of living, how long will your money last?**

**Fibonacci (1170-1250)**

**He discovered the technique we still use today for computing present and discounted values.**

## Interest Rates, Italy (1200–1400)




## Equation #1

$$t = \frac{1}{r} \ln\left[\frac{c}{c - Wr}\right]$$

## Equation #1

- Here is an example so you can see how to **use** the equation.
- You have $300,000 invested, earning 3% and you want to spend $30,000 per year.

## Equation #1

$$t = \frac{1}{3\%} \ln\left[\frac{\$30{,}000}{\$30{,}000 - (\$300{,}000 \times 3\%)}\right]$$

## Equation #1

$$t = \frac{1}{3\%} \ln\left[1.42857\right]$$

## Equation #1

$$t = \frac{1}{3\%}\left[0.35667\right]$$

# t = 11.9 years

**In how many years will the money run out?**
**Assuming a real discount rate of 1.5%**

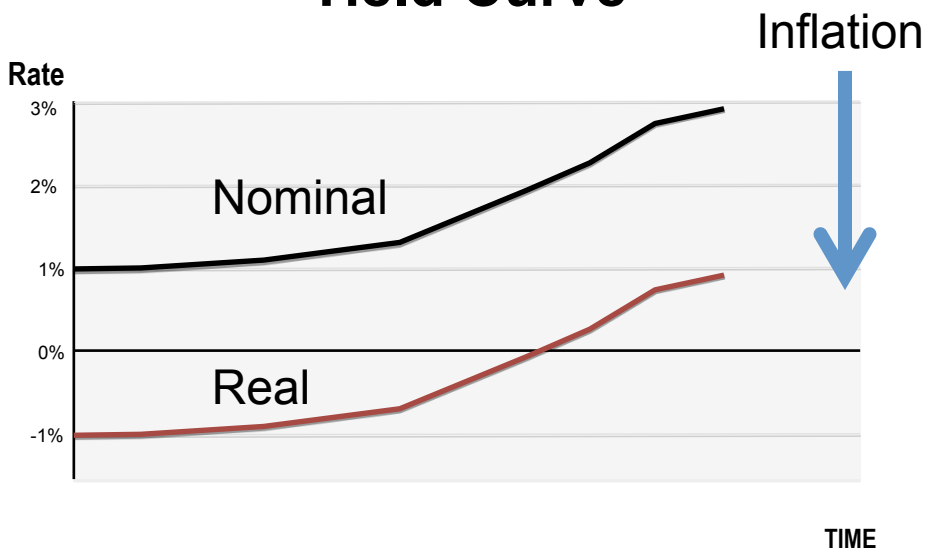
| Spending Rate (c) | Nest Egg at Retirement (W) | | |
|---|---|---|---|
| @ 1.5% | $200,000 | $300,000 | $400,000 |
| $20,000 | 10.8 | 17.0 | 23.8 |
| $25,000 | 8.5 | 13.2 | 18.3 |
| $30,000 | 7.0 | 10.8 | 14.9 |
| $35,000 | 6.0 | 9.2 | 12.5 |

## U.S Treasury Yield Curve Rates

| | NOMINAL | REAL (TIPS) |
|---|---|---|
| 5 year | 0.82% | -1.24% |
| 7 year | 1.34% | -0.84% |
| 10 year | 1.96% | -0.30% |
| 20 year | 2.73% | 0.44% |
| 30 year | 3.12% | 0.73% |

Source: http://www.treasury.gov/resource-center
Accessed April 27th, 2012

## Yield Curve



## Speaking of discount rates:

## What are the numbers used by public pension plans?

## Nominal Discount Rates for Public Pension Promises

| "Assumed" Number | Percent of Plans |
|---|---|
| 8.5% or Greater | 9% |
| Between 8.0% and < 8.5% | 48% |
| Between 7.5% and < 8.0% | 37% |
| Between 7.0% and < 7.5% | 6% |
| TOTAL… | 100% |

**Source**: *Wall Street Journal*, March 14, 2012, pg. C2
From Universe of 96 plans in Public Fund Survey

# How long will my number last?

**Not as long as you think.**

**Conversation #2:**
**How long will you spend living in retirement and how random is that number?**

- **British Demographer and Actuary**
- **Fellow of the Royal Society**
- **Never attended university!**
- **Brother-in-law of M. Montefiore**

## Benjamin Gompertz (1779-1865)

## How Many Years Beyond Retirement?

- Arithmetic Mean: = **19.7** years
- Standard Deviation: = **11.0** years

| Mon. | Tue. | Wed. | Thu. | Fri. |
|---|---|---|---|---|
| 20.8 | 2.3 | 12.1 | 9.0 | 34.2 |
| 4.3 | 20.1 | 30.3 | 27.5 | 23.4 |
| 20.7 | 4.7 | 11.4 | 20.4 | 35.6 |
| 4.9 | 20.3 | 30.2 | 30.2 | 33.9 |
| 34.5 | 29.1 | 19.6 | 20.9 | 28.8 |
| 18.8 | 24.3 | 18.7 | 19.9 | 28.3 |
| 21.0 | 30.2 | 40.5 | 34.2 | 17.0 |
| 10.3 | 6.2 | 4.2 | 24.0 | 17.5 |
| 8.5 | 36.3 | 24.7 | 11.8 | 0.3 |
| 26.3 | 5.9 | 26.2 | 19.2 | 14.4 |

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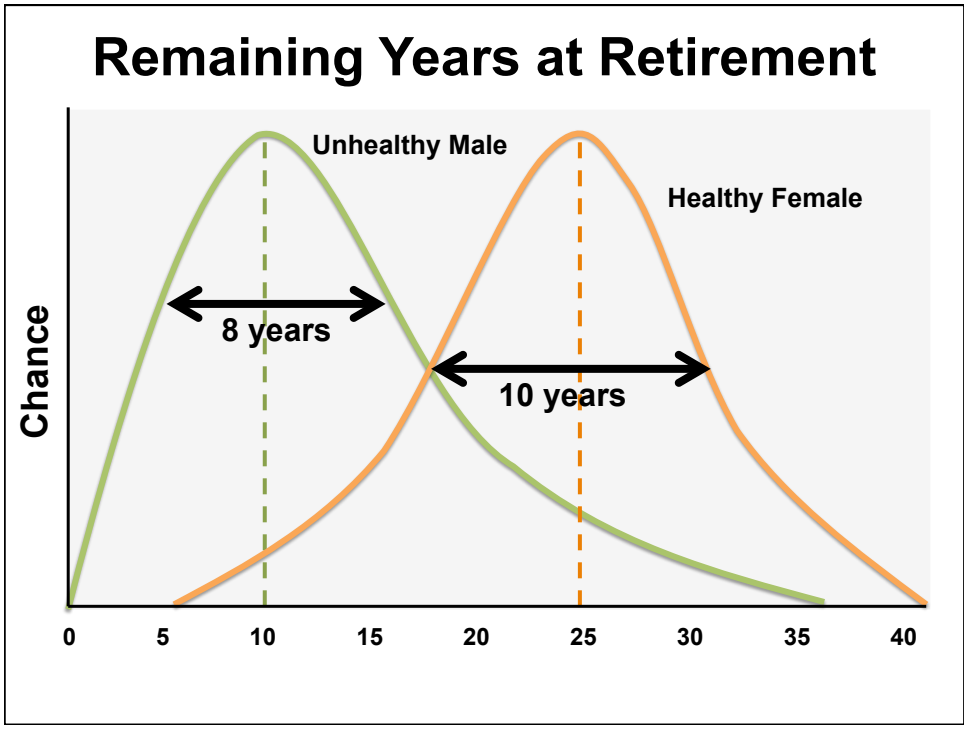
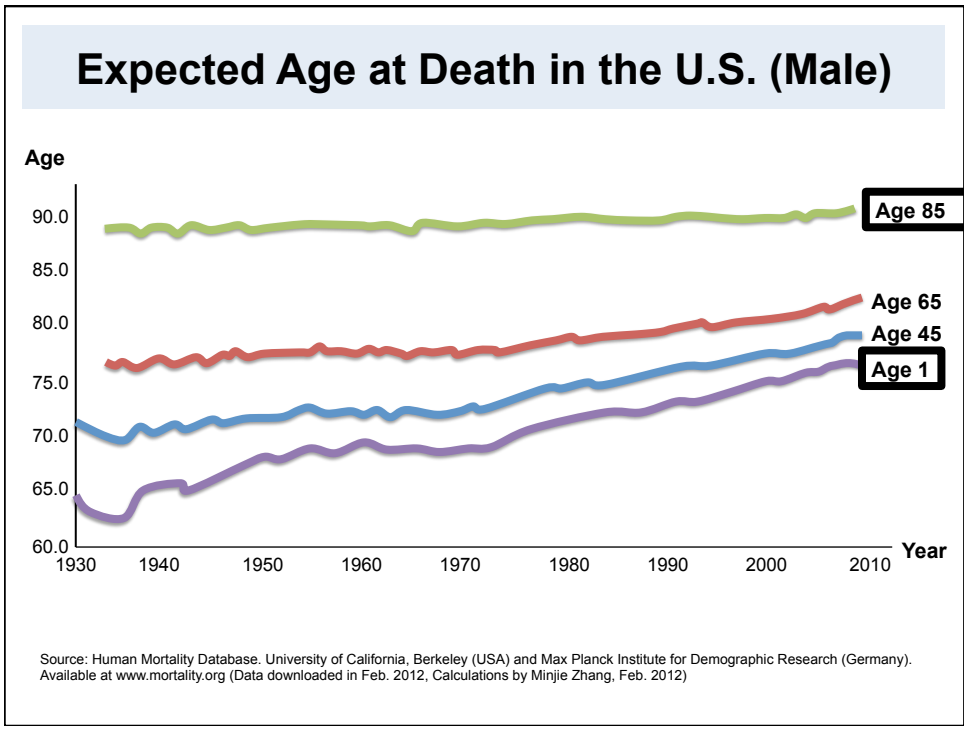
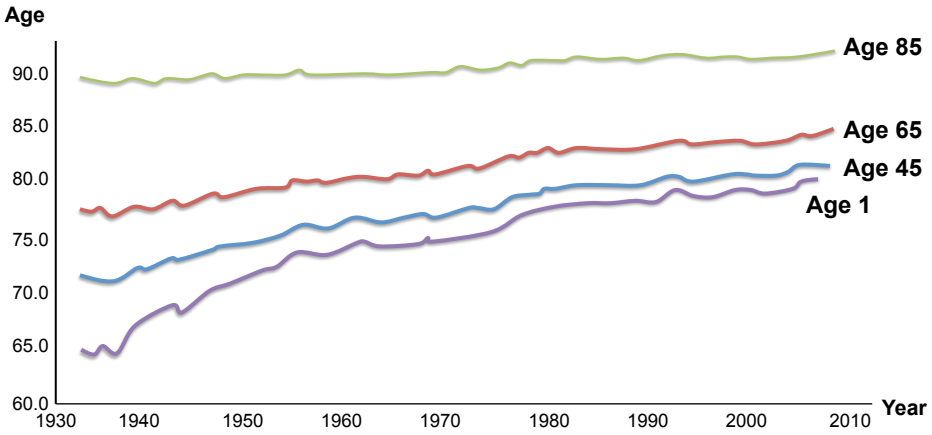


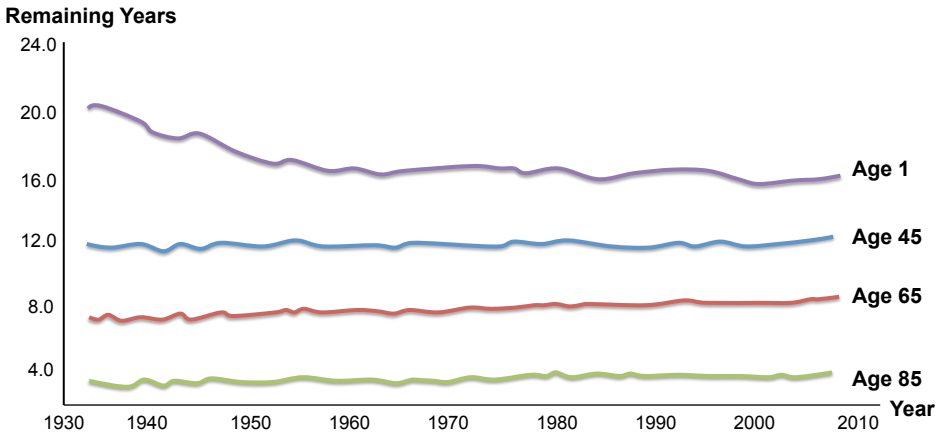



## Expected Age at Death in the U.S. (Female)




Source: Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org (Data downloaded in Feb. 2012, Calculations by Minjie Zhang, Feb. 2012)

## Standard Deviation of Remaining Life in U.S. (Male)

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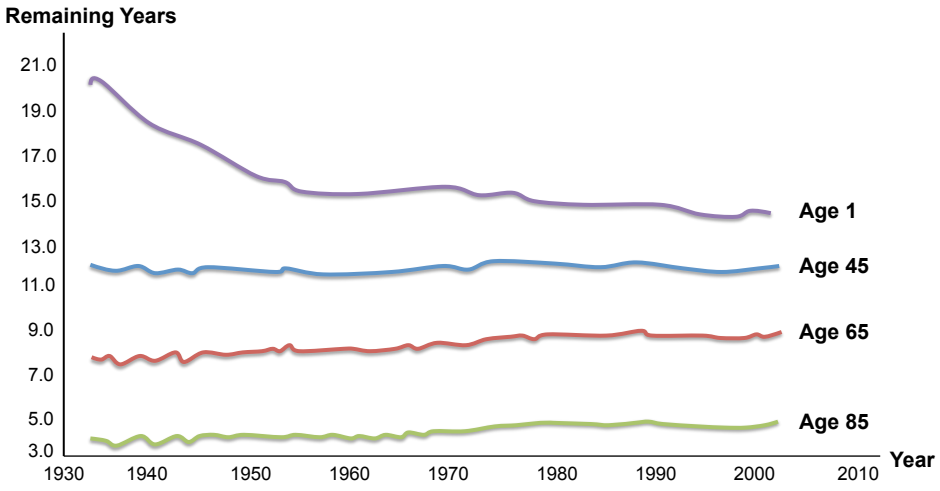
## Standard Deviation of Remaining Life in U.S. (Female)

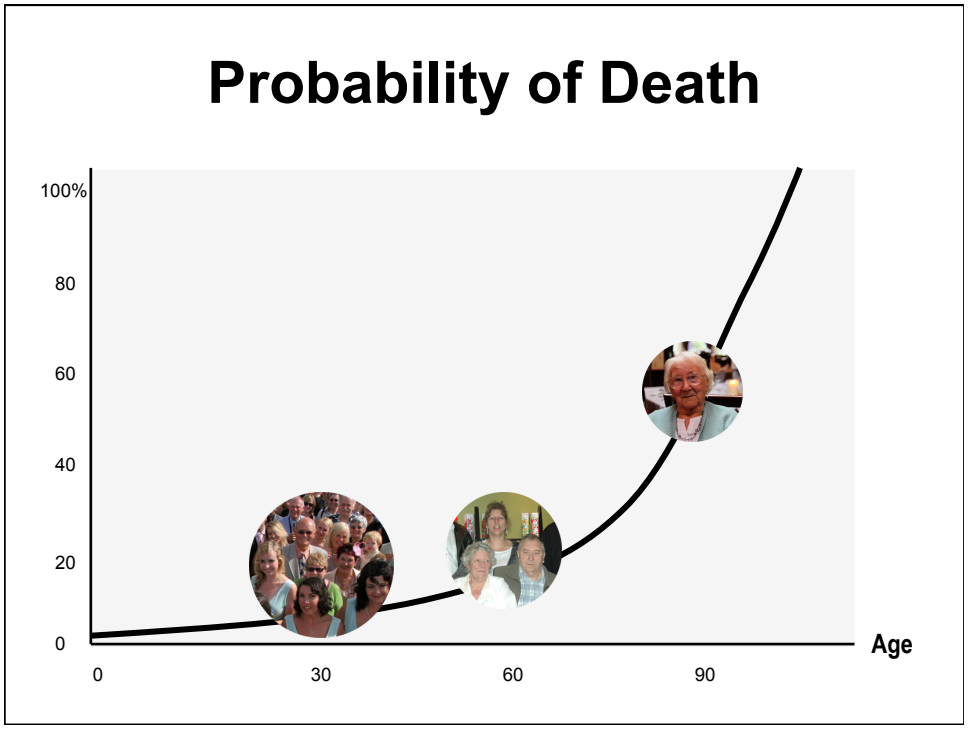


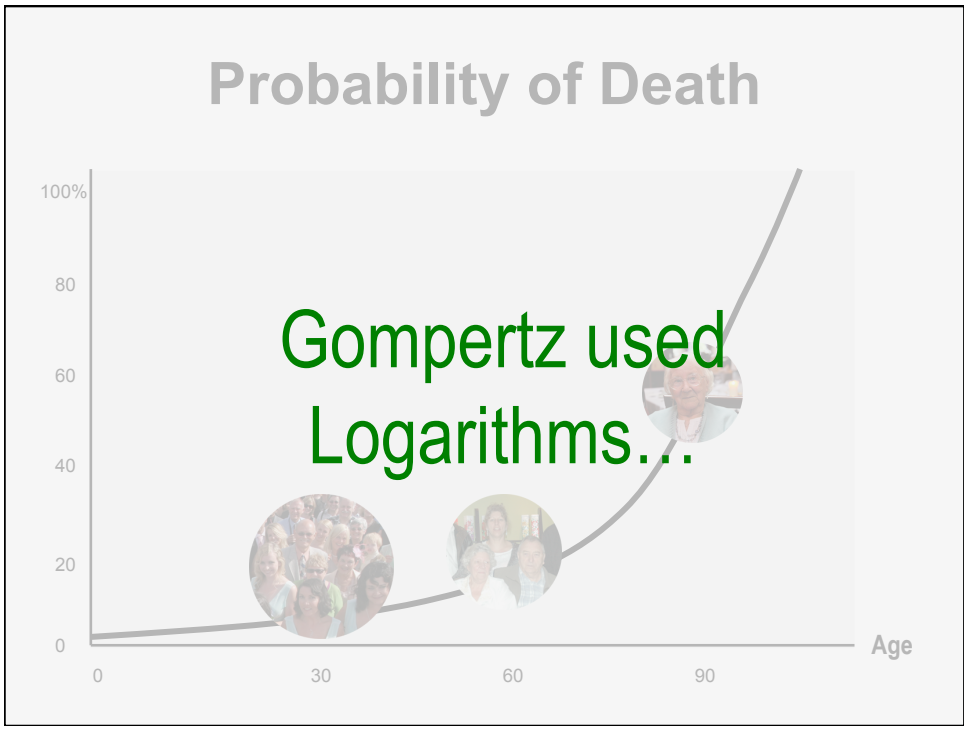

Source: Human Mortality Database. University of California, Berkeley (USA) and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org (Data downloaded in Feb. 2012, Calculations by Minjie Zhang, Feb. 2012)

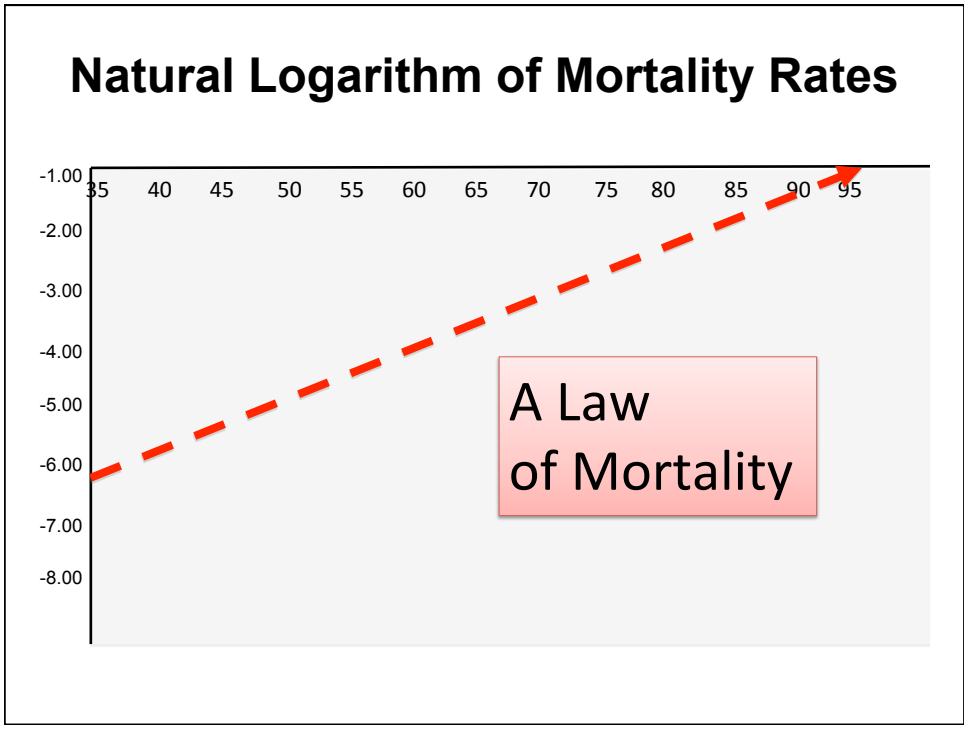
# Longevity in the U.S.

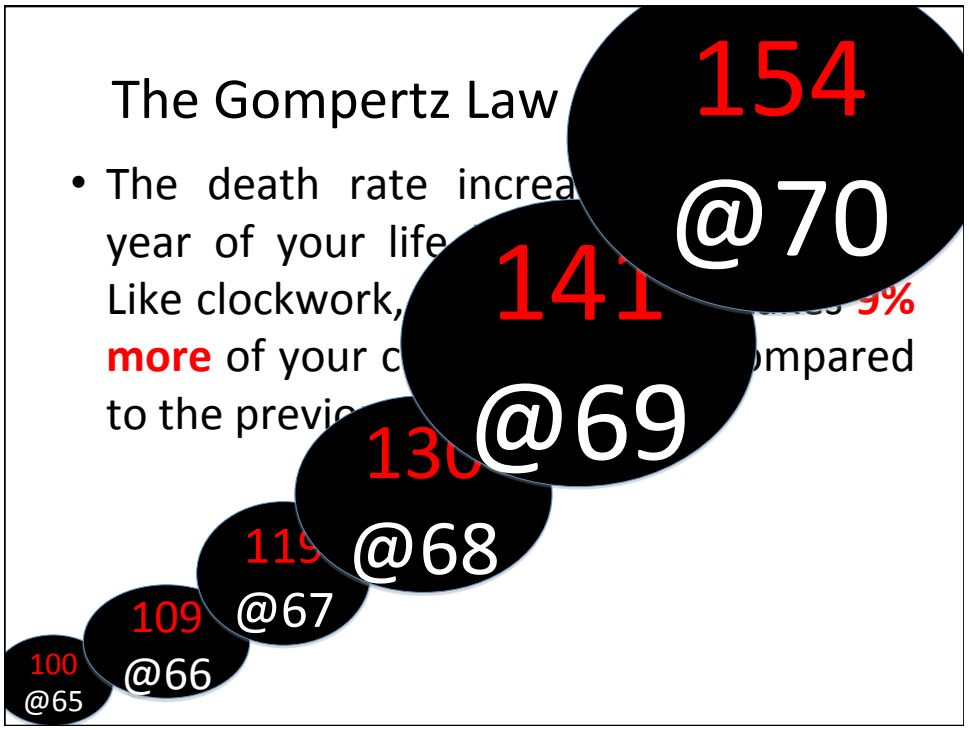
| Number of Americans… | |
|---|---|
| **> Age 90** | **2,000,000** |
| **> Age 100** | **97,000** |

*Source: Census Bureau as reported in Sep/ 20/2010 issue of Investment News*

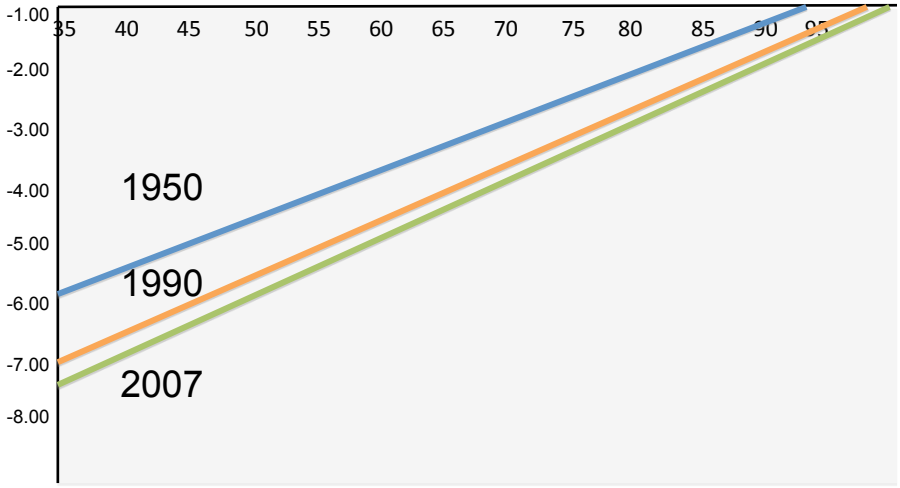




## Natural Logarithm of Mortality Rates: Cohorts over Time




THE NUMBERS GUY | March 2, 2012, 7:00 p.m. ET

## Death Gets in the Way of Old-Age Gains

Article | Comments (19)

Email Print Save Like 26 1 Tweet 23

A new research paper, and a census surprise, are calling into question some long-held beliefs about a morbid bit of math: how much mortality rates increase with age.

It's no surprise that the older a group of people get, the higher the percentage of them who will die in any given time period. Benjamin Gompertz, a 19th-century British mathematician, charted the increase in mortality rates as very regular. His Gompertz law of mortality says that each additional period brings a constant percentage increase in mortality rates.

## Equation #2

$$\ln[p] = (1 - e^{\frac{t}{b}})\, e^{\frac{x-m}{b}}$$

2.7183

## Equation #2

- You are x = 57 years old. The modal value of life is m = 87.25 years, and the dispersion coefficient is b = 9.5 years.
- What is the probability you will live for t = 33 more years, to the age of 90?

## Equation #2

$$\ln[p] = (1 - e^{\frac{33}{95}})\, e^{\frac{57 - 87.25}{95}}$$

$$\ln[p] = -1.29427$$

$$e^{\ln[p]} = e^{-1.29427}$$

$$p = (2.7183)^{-1.29427}$$

$$p = 0.2741$$

# 27.4%

**The probability a 57-year-old will live to the age of 90, under the given modal and dispersion value**

## Under One Law of Mortality

| Your Current Age | Probability of Living to 90 |
|---|---|
| 45 | 26.6% |
| 65 | 29.0% |
| 85 | 57.9% |

## Health and Wealth: Life Expectancy at Age 70

| Income Percentile | Healthy Male | Healthy Female |
|---|---|---|
| 20th | 8.2 yrs | 13.8 yrs |
| 40th | 9.1 yrs | 14.8 yrs |
| 60th | 10.1 yrs | 15.9 yrs |
| 80th | 11.2 yrs | 17.0 yrs |

**Source**: Federal Reserve Bank of Chicago, WP 2005-13 (De Nardi, French and Jones)

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## Impact of Education: Age 65 – 75
Increase / Reduction in Mortality Rate

| Education | Male | Female |
|---|---|---|
| < High School | +23% | +26% |
| High School | -2% | -9% |
| Some College | -10% | -19% |
| College | -38% | -32% |

Source: J.P. Cristia, August 2007, *Congressional Budget Office*, Working Paper #11
"The Empirical Relationship Between Lifetime Earnings and Mortality"

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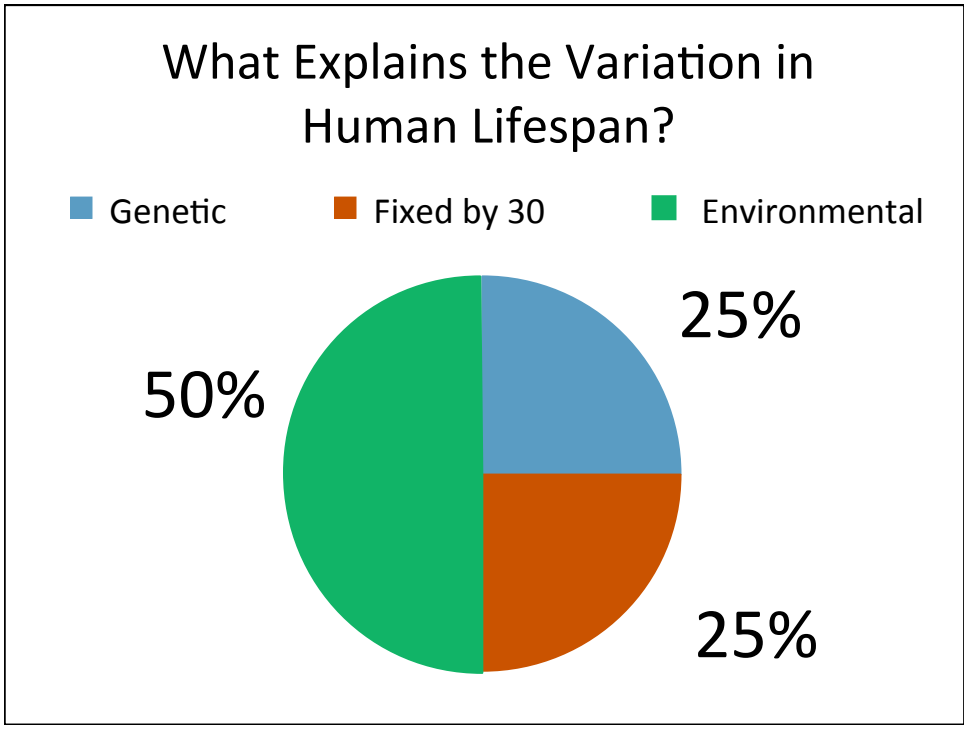



# How long will I spend in retirement?

**Longer than you think!**

## Conversation #3:
## What is your pension annuity really worth?

## And, how much would it cost to buy some more lifetime income?

- British Astronomer Royal
- Savilian Professor of Geometry at Oxford University
- Mapped earth's magnetic field
- Isaac Newton's *Principia* publisher

Wrote and published hundreds of papers on astronomy and geophysics, and one paper in 1693 on pricing life annuities!

## What Does Retirement Cost?
## $1,000 Monthly Income Starting at Age 65

How much do you need at age 65 to generate $1,000 per month during retirement?

## What Does Retirement Cost?
## $1,000 Monthly Income Starting at Age 65

| | Plan to Age | REAL Inflation-adjusted Investment Return | | | |
|---|---|---|---|---|---|
| | | 0.0% | 1.5% | 4.0% | 6.5% |
| Life Expectancy (50th Percentile): | 84.2 | $230,490 | $200,300 | $160,900 | $131,600 |
| 75th Percentile of Remaining Life: | 90.1 | $301,700 | $251,300 | $190,300 | $148,600 |
| 95th Percentile of Remaining Life: | 97.1 | $385,100 | $305,600 | $216,900 | $161,700 |
| Cost of REAL Pension Annuity March 2012 | Male | | $253,000 | | |
| | Female | | $281,000 | | |
| Cost of REAL Pension Annuity March 2009 | Male | | $183,000 | | |
| | Female | | $199,000 | | |

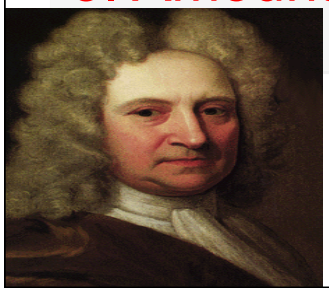
Retirement is more expensive than ever before….perhaps we should consume less of it.

*An Eſtimate of the Degrees of the* Mortality *of Mankind, drawn from curious* Tables *of the* Births *and* Funerals *at the City of* Breſlaw ; *with an Attempt to aſcertain the Price of* Annuities *upon* Lives. *By Mr.* E. Halley, *R.S.S.*

THE Contemplation of the *Mortality* of *Mankind*, has beſides the *Moral*, its *Phyſical* and *Political* Uſes, both which have been ſome years ſince moſt judiciouſly conſidered by the curious Sir *William Petty*, in his *Natural* and *Political* Obſervations on the Bills of *Mortality* of *London*, owned by Captain *John Graunt*. And ſince in a like Treatiſe on the Bills of *Mortality* of

## Three ingredients to value a pension annuity:

1. Survival probability to all ages
2. Present value interest factor
3. Amount of periodic cash flow



## Equation #3

$$a_x = \sum_{i=1}^{\infty} \frac{{}_iP_x}{(1+R)^i}$$

*"...People always live for ever when there is an annuity to be paid them..."*

JANE AUSTEN (1811)
***Sense and Sensibility***

## Value of a pension annuity

Guaranteed $1 per year for the rest of your life

| Interest Rate \ Age | 55 | 65 | 75 | 85 |
|---|---|---|---|---|
| 1.0% | $24.03 | $17.35 | $11.23 | $6.32 |
| 3.0% | $18.23 | $14.07 | $9.70 | $5.77 |
| 5.0% | $14.34 | $11.67 | $8.48 | $5.29 |
| 8.0% | $10.59 | $9.13 | $7.07 | $4.69 |

## Halley's Equation: Usage and Implications

- Take lump-sum or pension annuity?
- If long-term interest rates go back to normal in 3-5 years, how much more annuity income can I expect?
- At what age should I start my Social Security or DB pension?
- Is it better to protect my spouse with life insurance or with a joint-life annuity?
- What is the value of a GLWB rider?

## What is a true inflation-adjusted lifetime of income really worth?

**Getting close to infinity!**

**Conversation #4:**
**What is a proper spending rate from your nest egg, and what financial and economic factors does it depend on?**

Survey by *Investment News* on withdrawal rates used by advisors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24 InvestmentNews | January 23, 2012

## Rethinking 4% withdrawal rule

Continued from Page 1

time to be conservative and prepare for a possibility that we may enter a period of time that's even worse than the 1970s," he said.

Mr. Bengen readily admits that the 4% rule isn't perfect.

For one thing, it starts to unravel during periods of high inflation, when withdrawal rates can quickly escalate to unrealistic levels.

Also, the rule is based on a buy-and-hold strategy, which "is nuts in this environment," Mr. Bengen said.

"You get zero returns by just holding," he said. "I try to get my clients out of the market when it's expensive and get them in when it's cheap, rather than just sitting there and getting beat by the market."

But the buy-and-hold component

**Deserves credit!**

**William P. Bengen:** "I haven't seen anything yet that says this rule is going to fail."

## Talking with William P. Bengen

*William P. Bengen is the father of the 4% withdrawal rule and president of fee-only Bengen Financial Services Inc.*

**Q. What led you to work on optimal withdrawal rates?**

**A.** I started the research because, in the early days, I had clients looking at retirement 10 or 15 years down

or 2004. That brought the withdrawal rate up to 4.5%.

I didn't have a database for other asset classes that went back 80 years, and I never experimented with real estate investment trusts or commodities; I suspect you'll get diminishing returns [with more asset classes] after a certain point, so I decided not to push the envelope.

3% or 3.5% when they're starting out. That's painful. I'd rather start with a high withdrawal rate and then make adjustments later.

**Q. What do you say to clients who say a 4% withdrawal is insufficient?**

**A.** I had had a couple of clients for whom it was a constra

# Understanding the 4% Rule…

This year… $100 **Spend $4** $96.0

-16.67%

Next Year… $80.0 **Spend $4 + CPI**

**Not 4% of $80 = $3.20**

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But, economists and financial practitioners have different views on spending rates.



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**Irving Fisher (1867-1947)**

- Professor of Economics, Yale.
- Created first inflation-indices.
- Inventor, entrepreneur, spokesperson, health advocate.
- Best known for his infamously incorrect forecast of the stock market in 1929.

Irving Fisher is most famous amongst economists for the following insight:

***Real*** *Return =* ***Nominal*** *Return*
*minus Inflation*

THE
THEORY OF INTEREST
As Determined by
IMPATIENCE
To Spend Income
OPPORTUNITY
To Invest It

BY
IRVING FISHER
PROFESSOR OF ECONOMICS, YALE UNIVERSITY

To understand Irving Fisher's insight on consumption rates, lets play the Retirement Income **60 by 60** Board Game.

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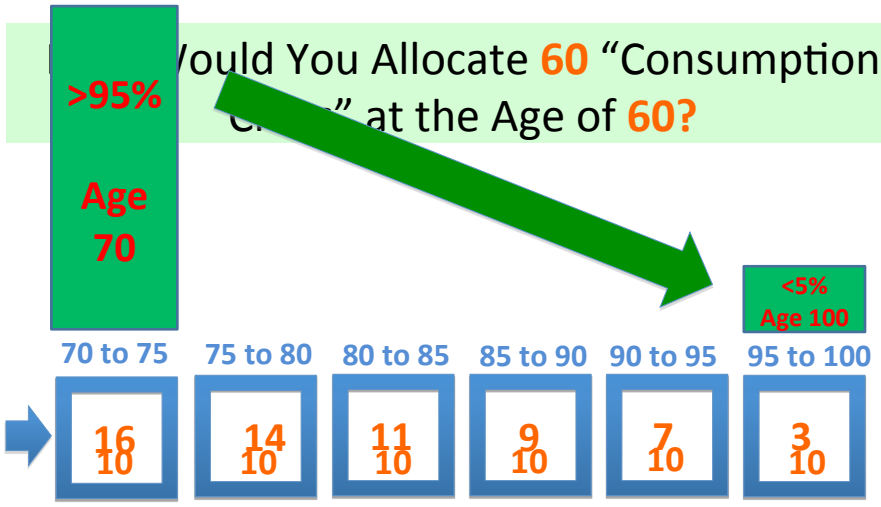


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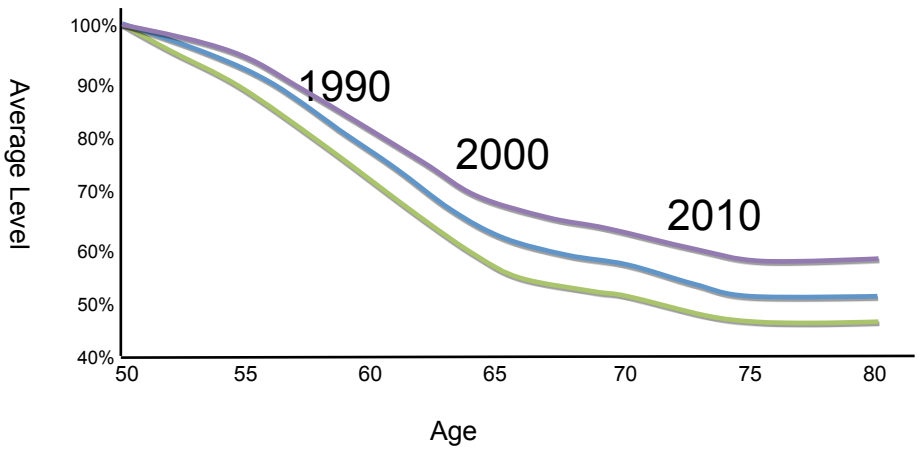
## Irving Fisher (1930)
## *The Theory of Interest*

*"The shortness of life thus tends powerfully to increase the degree of impatience or rate of time preference beyond what it otherwise might be."*

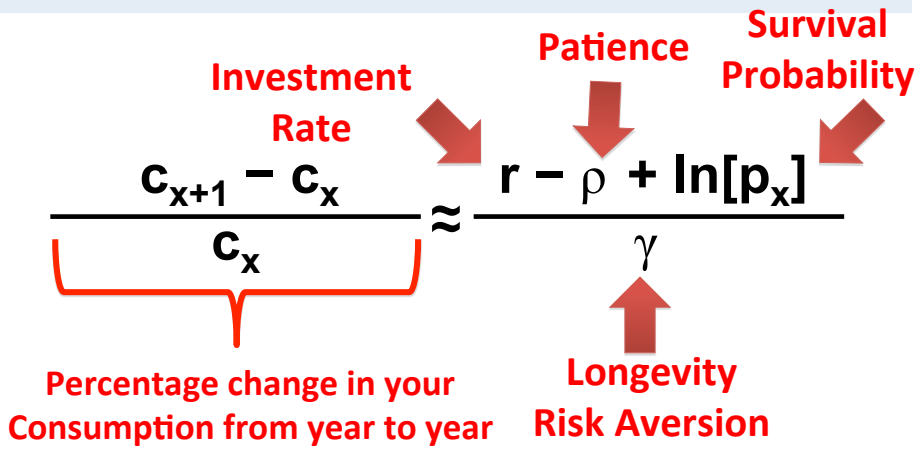
*"Everyone at some point in his life doubtless changes his degree of impatience for income."*

*"He expects to die and he thinks: Instead of pilling up for the remote future, why shouldn't I enjoy myself during the few years that remain."*

**Retirement Expenditures by Retiring Population: Baseline at Age 50**



## Equation #4




## Optimal Spending Rates from $100 at age 65

*Realistic Investment Assumption: 2.5% Real*

| | Increasing Longevity Risk Aversion…. | | |
|---|---|---|---|
| **Pre-Existing Pension Annuity** | **Low (1)** | **Med. (2)** | **High (8)** |
| **$0** | **6.33%** | **5.30%** | **4.12%** |
| **$1** | **6.80%** | **5.65%** | **4.32%** |
| **$2** | **7.16%** | **5.92%** | **4.48%** |
| **$5** | **8.02%** | **6.55%** | **4.83%** |

**Note: Assumes 5% Survival to Age 100, 25% Survival to Age 93 and 50% to Age 87. Subjective Discount Rate (ρ) assumed equivalent to real investment rate.**

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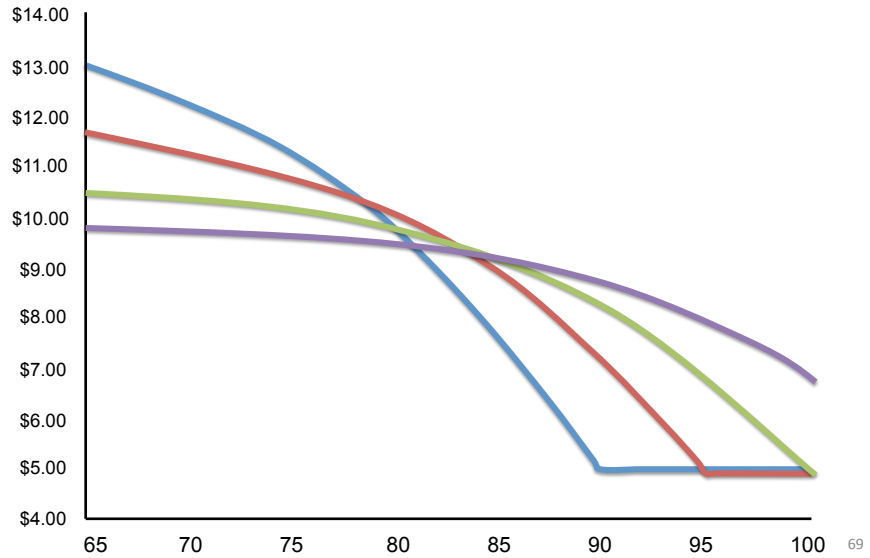
## Optimal Consumption

$5 Pension Income with Real Rate = 2.5%



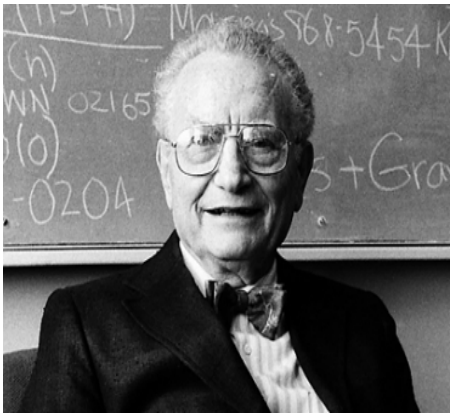

## What is the right retirement nest egg spending rate?

**In addition to your views on the market, it depends on pension income and longevity risk aversion.**

## Conversation #5:

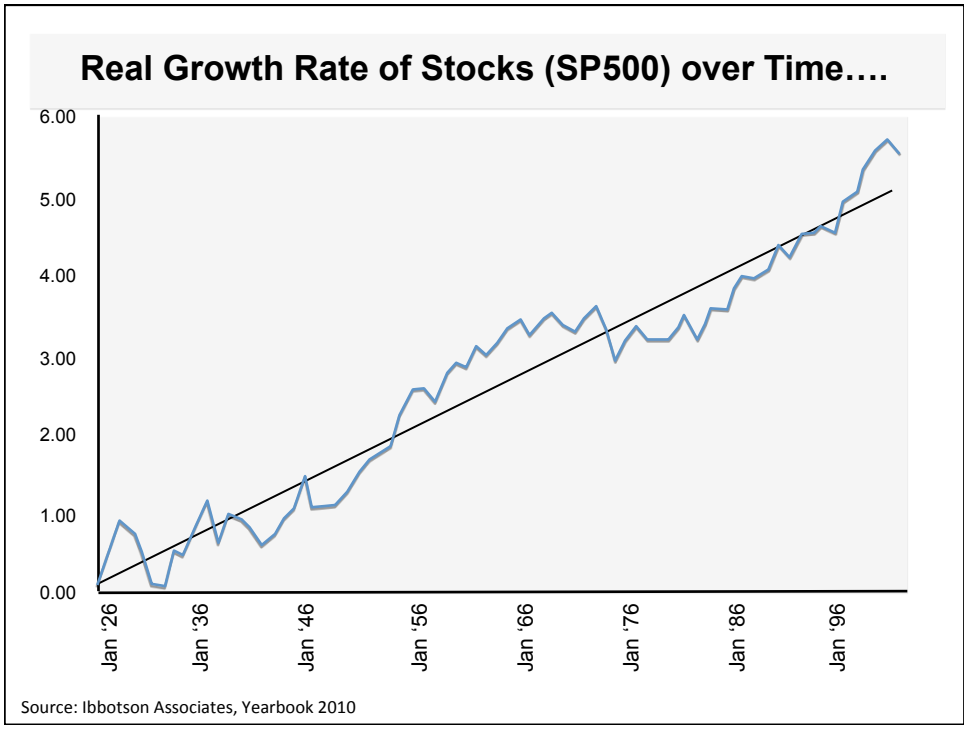
**What is the proper mix of risky stocks vs. safer bonds as you age?**

**What should your asset allocation depend on, besides risk tolerance?**



- Professor of Economics MIT
- Nobel Laureate 1970.
- Economic Advisor to J.F.K.
- Many of his students went-on to win Nobel prize.
- Author of most popular textbook in economics.
- Born in Gary, Indiana.

**Paul Samuelson (1915-2009)**

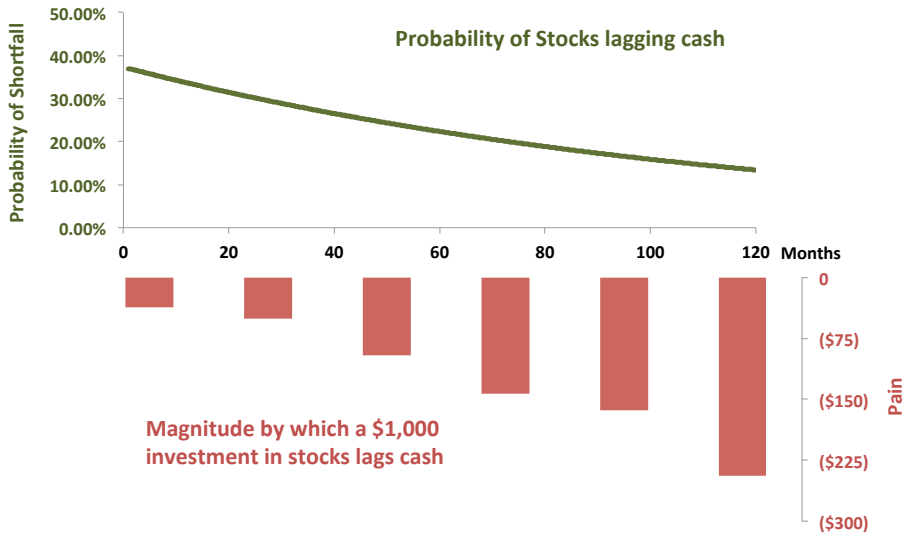



## $1,000 Invested over 1000 Months: Stocks Doing Worse than Cash

| Period | Probability | $ Pain |
|---|---|---|
| 1 month | 40% | 37. |
| 2 months | 39% | 51. |
| 6 months | 35% | 96. |
| 1 year | 31% | 143. |
| 5 years | 20% | 164. |

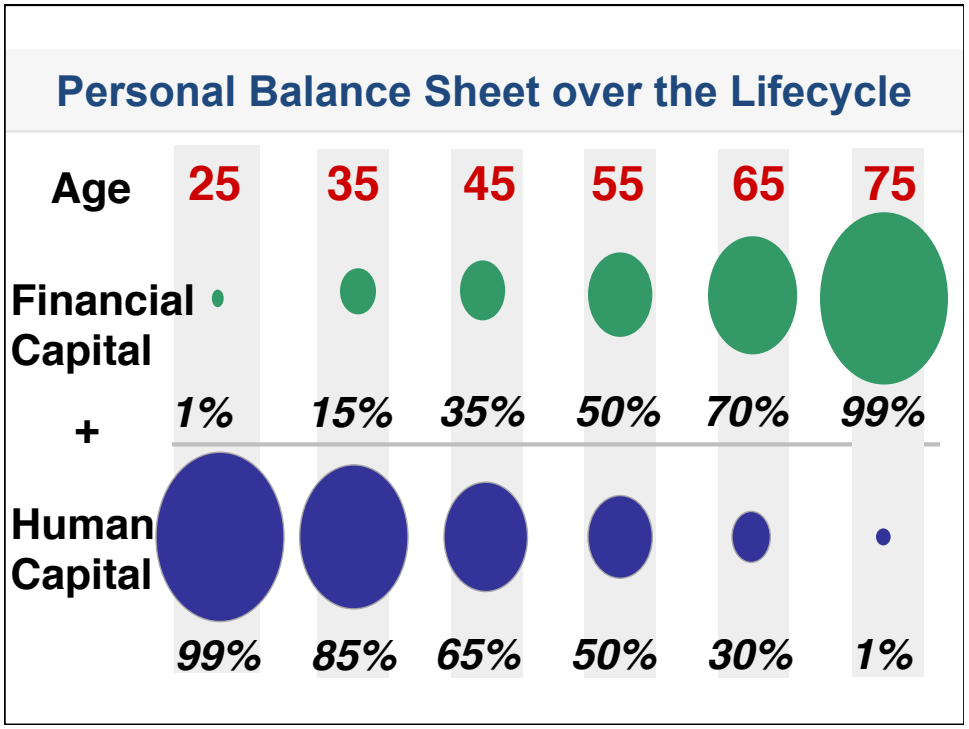
Sources: Federal Reserve Bank of St. Louis and Center for Research in Security Prices (March 2012) 74

## $1,000 Invested over 1000 Months: Stocks Doing Worse than Cash




## In the words of Paul Samuelson

*"…I do not favor or disfavor any changes in equity tolerance induced by lengthening of the investment horizon. What I argue is that a risk-averse person who is an expected utility of wealth maximizer, will not by any valid application of the law of large numbers have to be more equity tolerant when time is large…."*




## Equation #5

$$\Psi = \frac{1}{\gamma}(HC + FC)\left(\frac{\mu - r}{\sigma^2}\right)$$

## Numerical Example….

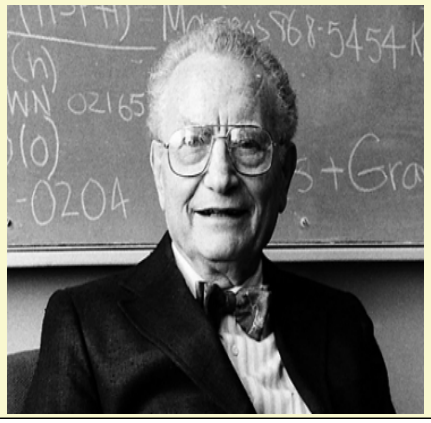
Estimated Human Capital = $500,000
Financial Capital = $500,000

Bonds Earn 2%.
Stocks Earn 8% with a volatility of 20%, which is a 6% growth rate.

$$\Psi = \frac{1}{\gamma}(\$1{,}500{,}000)$$

| Low risk aversion | Medium risk aversion | High risk aversion |
|---|---|---|
| $\Psi = 1$ | $\Psi = 3$ | $\Psi = 8$ |
| **$1,500,000** in stocks | **$500,000** in stocks | **$187,500** in stocks |

## As you age, how much in (risky) stocks vs. (safe) bonds?

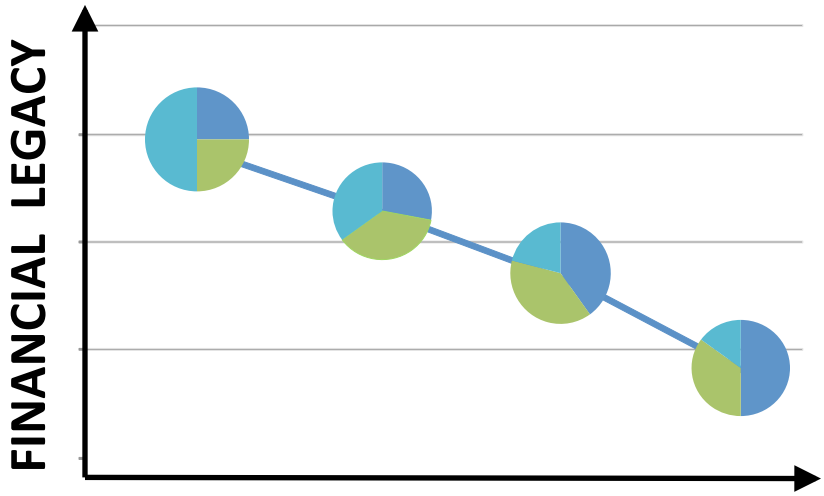


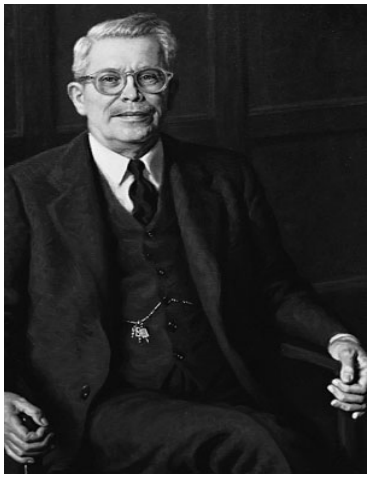
**Age is nothing but a number!**

## Conversation #6:

**How important is leaving a financial legacy and what is it really worth today?**

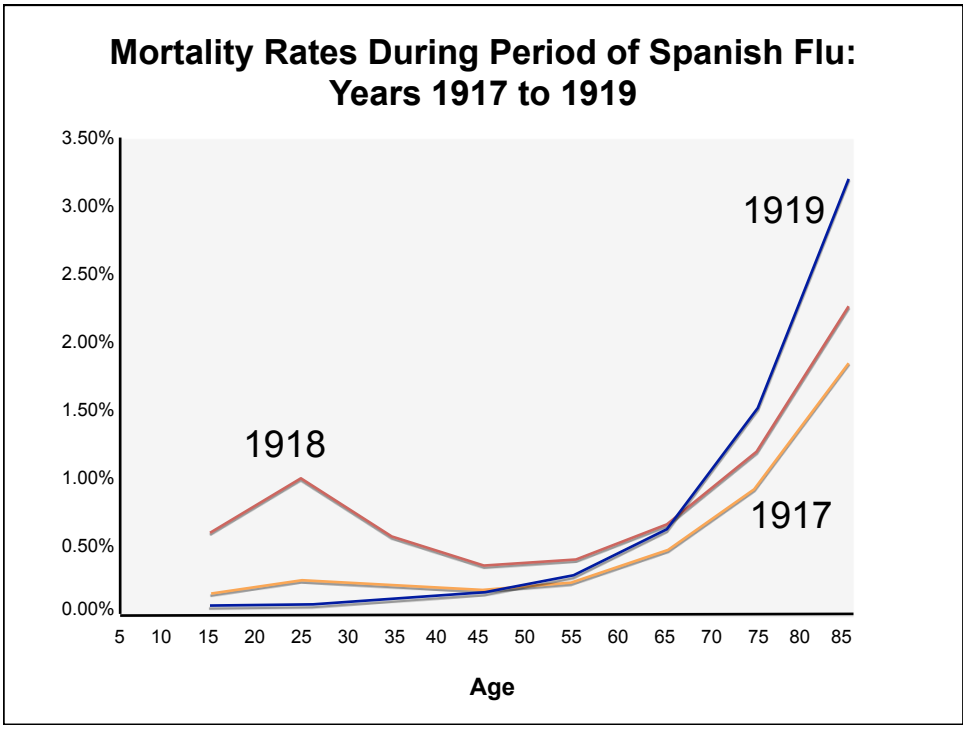
## ECONOMIC TRADEOFF AT RETIREMENT



- **Professor of Insurance, Wharton.**
- **Founder of the American College**
- **Promoted the concept of human life value (HLV)**
- **Consultant to the US Government**
- **Traveled the world giving lectures on the importance of life insurance.**

**Solomon S. Huebner**
**(1882-1964)**




## Equation #6

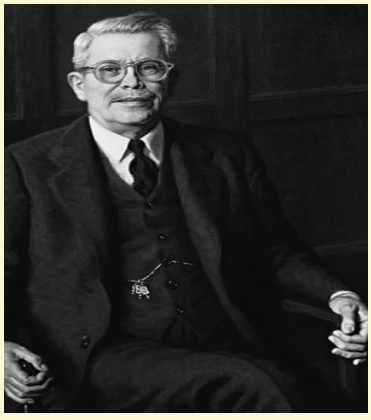
$$A_x = \sum_{i=0}^{\infty} \frac{({}_i p_x)(q_{x+i})}{(1+R)^{(i+1)}}$$

What is the net single premium (value) of a desired (promised) death benefit?

## Value of a $100,000 Death Benefit at Various Ages

| | Age | | | |
|---|---|---|---|---|
| Valuation Rate | 55 | 65 | 75 | 85 |
| 1.0% | $75,973 | $82,648 | $88,773 | $93,683 |
| 3.0% | $45,319 | $57,776 | $70,894 | $82,697 |
| 5.0% | $28,315 | $41,656 | $57,592 | $73,538 |
| 7.0% | $18,560 | $30,959 | $47,552 | $65,846 |
| Life Expectancy | 28.0 | 19.4 | 12.1 | 6.6 |

## How to fine-tune your financial legacy?



**Consider using all types of insurance**

## Conversation #7:

**Taking everything into account, if you continue on the current path, is your retirement income plan sustainable?**

**Andrey N. Kolmogorov (1903-1987)**

- Russian Mathematician
- Parents were communist revolutionaries.
- It is said: "*What Euclid did for geometry, Kolmogorov did for probability*."
- Awarded Order of Lenin seven (7) times.
- Founded schools for children to study math and sciences.

## Equation #7

Lifetime Ruin Probability (LRP)

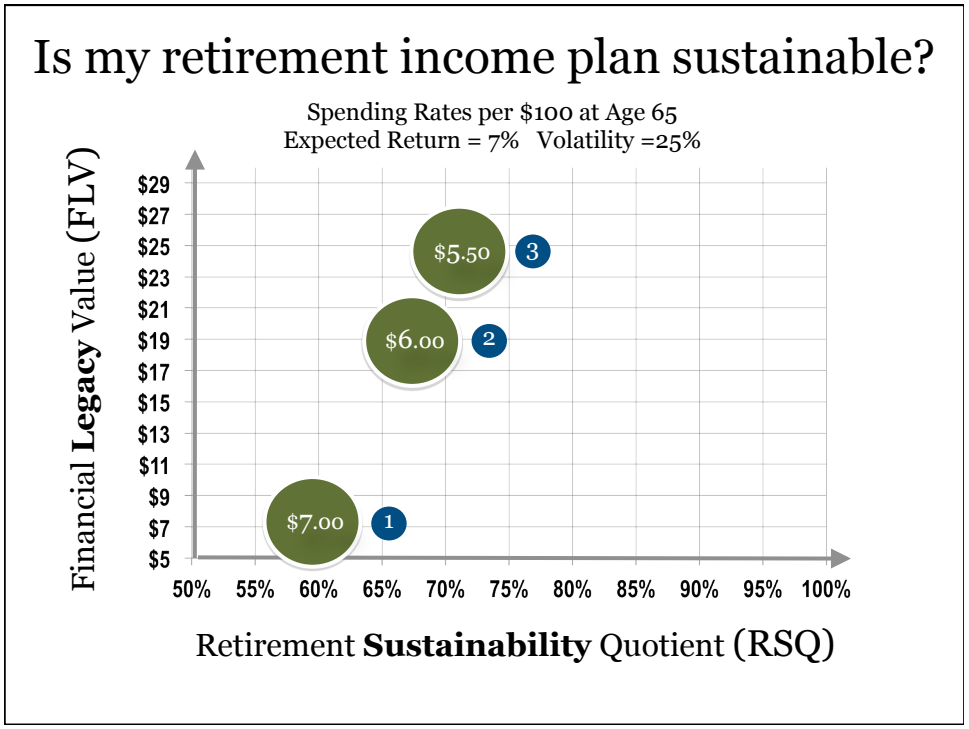
$$P\lambda_t = \frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} + \frac{1}{2}\sigma^2 w^2 \frac{\partial^2 P}{\partial w^2}$$

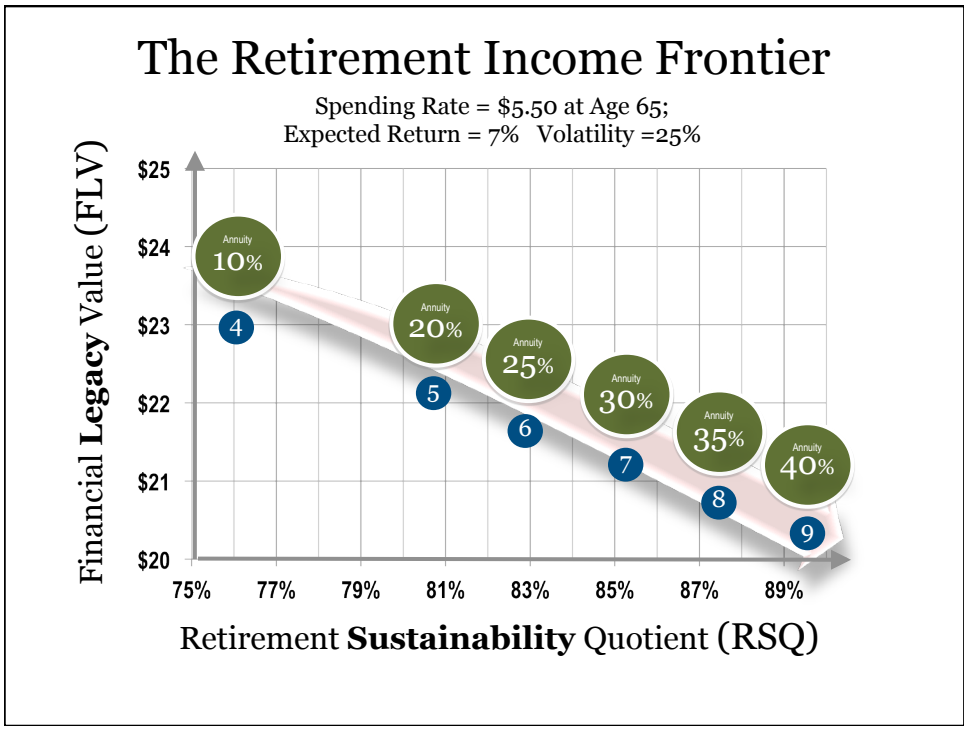
Will I run out of money,
before I run out of life?

## Lifetime Ruin Probability

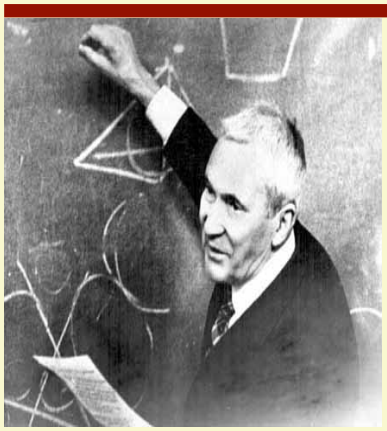
| Retire \ Spend | $4 per $100 | $6 per $100 |
|---|---|---|
| Age 65 | 7.6% | 22.1% |
| Age 75 | 2.5% | 9.8% |

Parameters: m = 87.25, b = 9.5, μ = 8%, σ = 20% (growth of 6%)



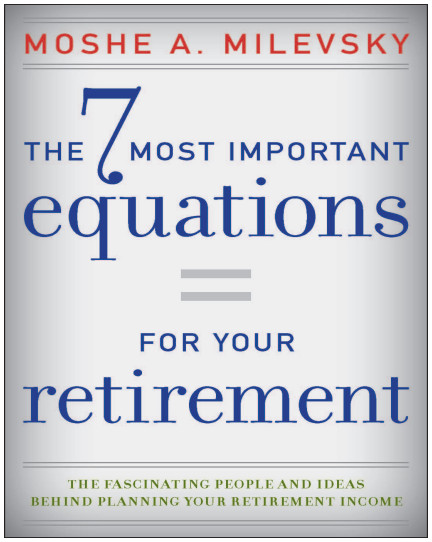



# Is your current plan sustainable?



## It is all about the probabilities.

## Conclusion:




1. Retirement income planning is more than just a "math problem"…but you can't avoid the numbers either.
2. There are some basic equations that all financial advisors should be aware of.
3. Reduce reliance on "black boxes" and move any debate to assumptions.
4. I hope you appreciate and hopefully enjoyed the brief history lesson…