Strategic Financial Planning over the Lifecycle

Chapter #13: Advanced Material Part I. Calculus of Variations

Narat Charupat, Huaxiong Huang and Moshe A. Milevsky

Ch. #13: Lecture Notes

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- assume that for $x \in (R, D)$ we have zero wage

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cont'd

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- **Note:**$(g - v)$ becomes a real (inflation-adjusted) quantity–we don't need to make guess about future inflation rates

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- **Take-away**: human capital in tomorrow's dollars might be larger than the value of human capital in today's dollars

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- to get net-human capital, we must subtract off the value of **implicit liabilities** from human capital

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- b_x –estimated cost
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- ignoring implicit liabilities: $s_t^* = w_t - c_t^*$

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- when $k = v$ the expression collapses to $w_x - c_x^*(D - x) = 0$

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- **EXAMPLE**
$x = 35$, $\mathbf{F}_x = \$100,000$ (financial capital), $w_{35} = 50,000$ p.a., $g = 6\%$ p.a., $R = 65$, $D = 95$, $b_{35} = \$20,000$ (minimum subsistent level of consumption), $\tilde{g} = 2\%$, $v = 5\% \Rightarrow \mathbf{H}_{35} = \$1,749,294$, $i\mathbf{L}_{35} = \$556,467$, $\mathbf{W}_{35} = \$1,292,827$

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and

$$\lim_{k \to v} c_x^* = \frac{\mathbf{W}_x}{D-x} \tag{19}$$

- **EXAMPLE**
$x = 35$, $\mathbf{F}_x = \$100,000$ (financial capital), $w_{35} = 50,000$ p.a., $g = 6\%$ p.a., $R = 65$, $D = 95$, $b_{35} = \$20,000$ (minimum subsistent level of consumption), $\tilde{g} = 2\%$, $v = 5\% \Rightarrow \mathbf{H}_{35} = \$1,749,294$, $i\mathbf{L}_{35} = \$556,467$, $\mathbf{W}_{35} = \$1,292,827$
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Effect of Wage Growth and Valuation Rates

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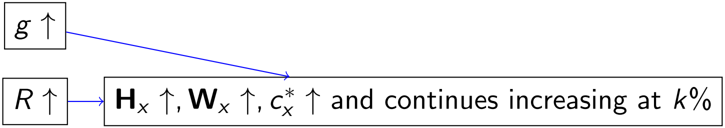

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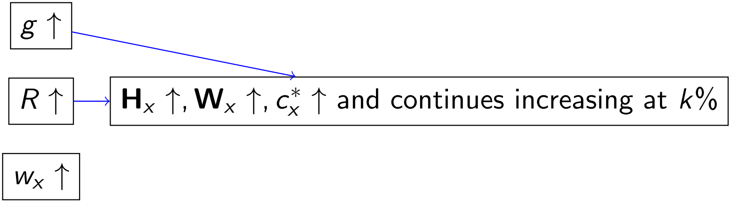

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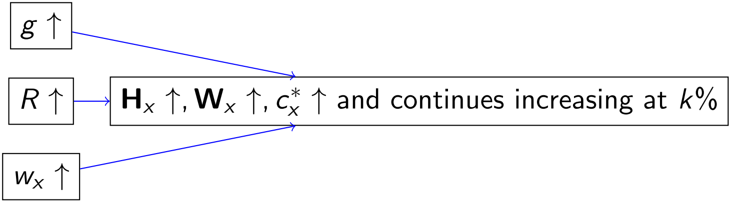

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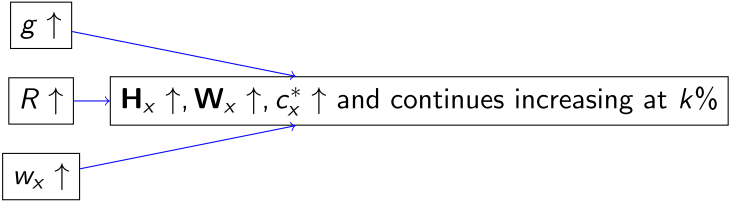

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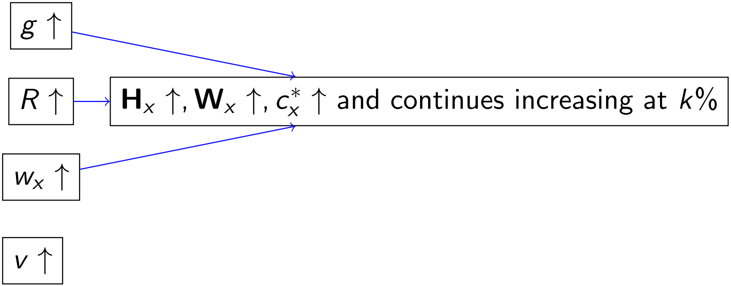

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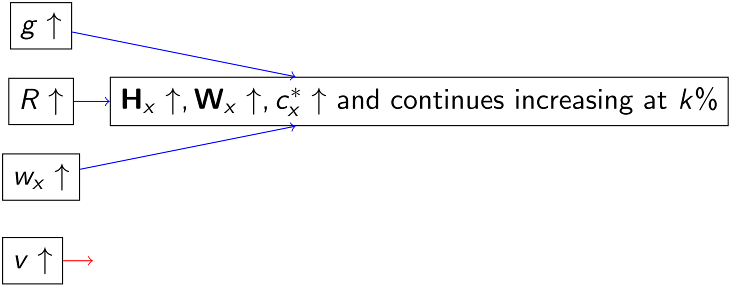

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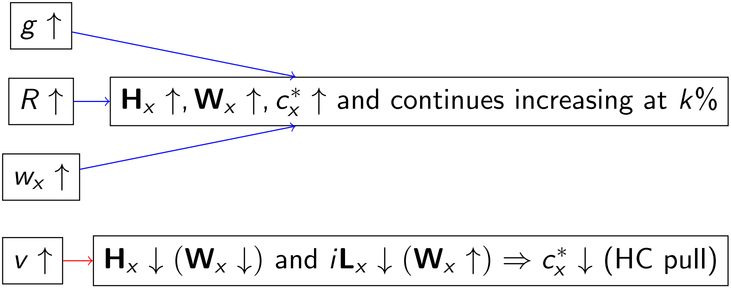

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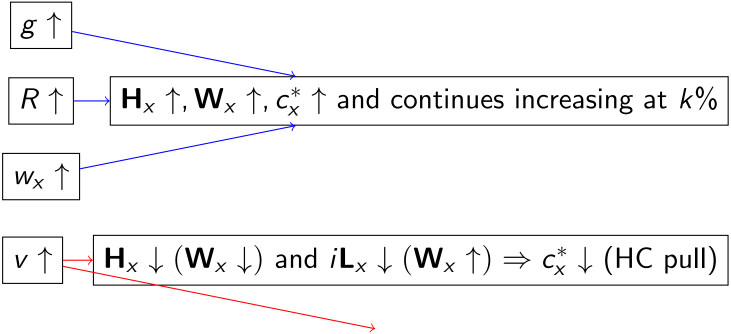

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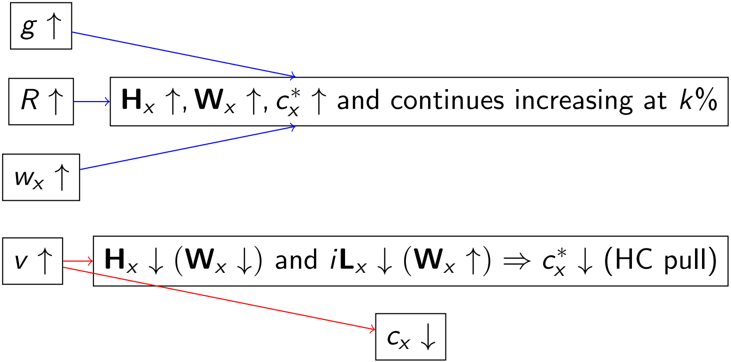

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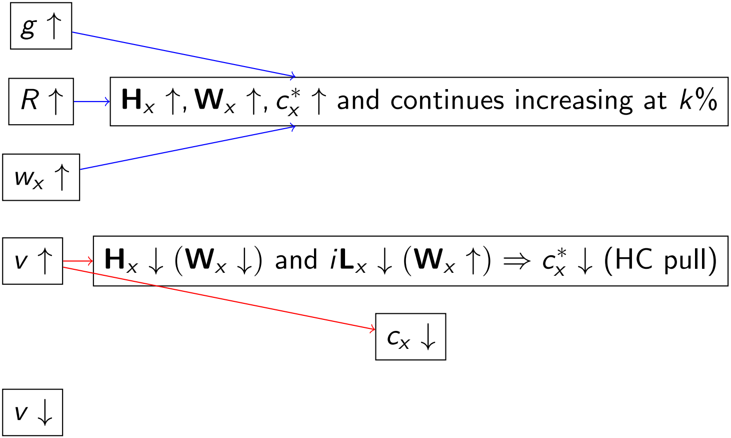

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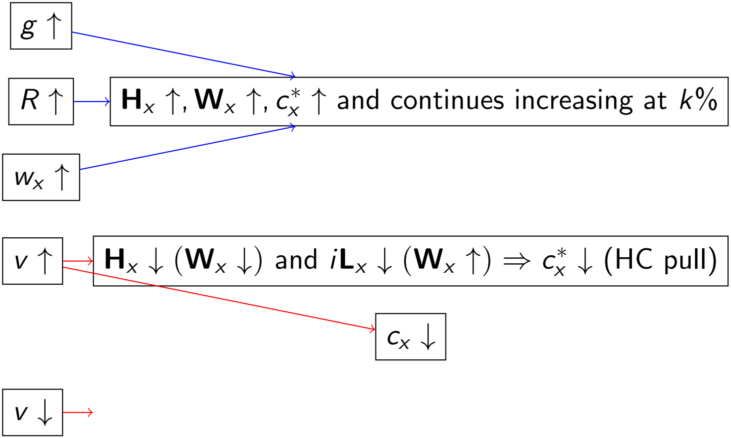

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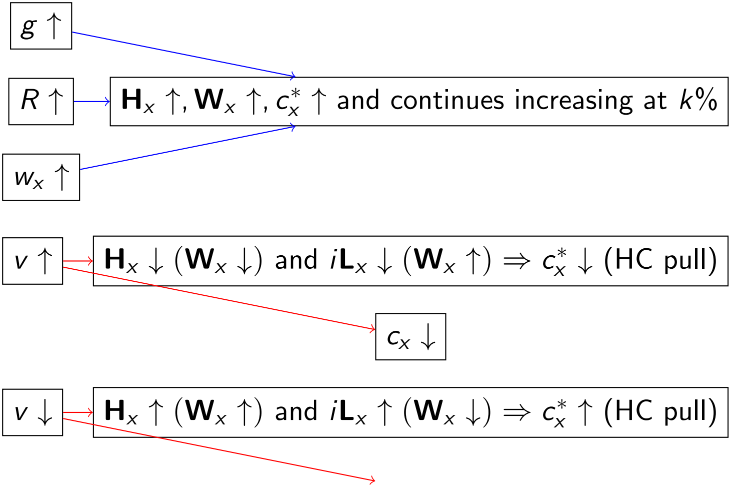

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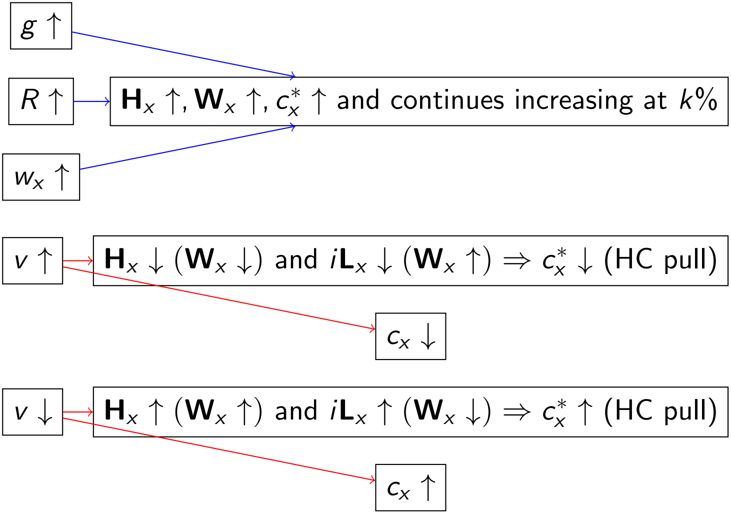

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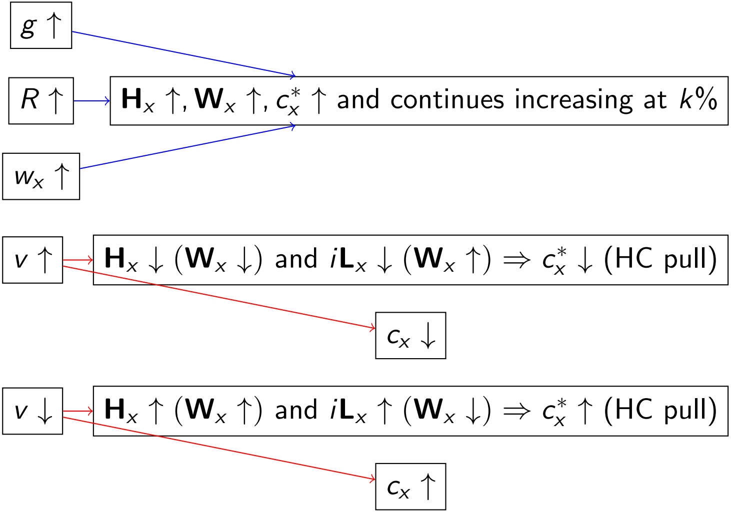

Note: g and k determine which effect dominates

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Calculus of Variation

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   where 2, 3 denote partial derivative w.r.t. 2^{nd} and 3^{rd} variables

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cont'd

$$
\begin{aligned}
\delta J &=^{IBP} \int_a^b \left\{ \left[\phi_2(t, z_t^*, \dot{z}_t^*) - \frac{d}{dt}\phi_3(t, z_t^*, \dot{z}_t^*) \right] \delta z_t + h.o.t. \right\} dt \\
&\quad + \phi_3(t, z_t^*, \dot{z}_t^*)\delta z_t \Big|_{z_a}^{z_b} \qquad (27) \\
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since z_a and z_b are given and $\delta z_a = \delta z_b = 0$

cont'd

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cont'd

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5. necessary condition for *optimality* is given by the Euler-Lagrange equation

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- assume a *relative risk-aversion* (CRRA) utility function

$$u(c_t) = \begin{cases} \frac{c_t^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1 \\ \ln(c_t) & \gamma = 1 \end{cases} \tag{32}$$

Solution of Optimal Consumption Problem

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- **Note**: for $\gamma \neq 1$ we will actually use $u(c_t) = c_t^{1-\gamma}/(1-\gamma)$ for simplicity as it does not affect the optimal solution

cont'd

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cont'd

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cont'd

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- **Assumptions**:

  a) borrowing rate = lending rate =constant

  b) no pension after retirement

  c) no mortality risk

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- as long as ${}_t p_x$ is constant or decreasing w.r.t. t

$$ {}_t p_x = e^{-\int_x^{x+t} \lambda_s ds} \tag{41} $$

  λ_s: instantaneous rate of death at age s

From $F_x \to f_x(t) \to \lambda_{x+t}$ and back again

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- we obtain the density function:

$$ f_x(t) = \frac{\partial}{\partial t}(1 - {}_t p_x) = (1 - F_x(t))\lambda_{x+t} \tag{43} $$

cont'd

- use equation (43) to represent the **Instant Force of Mortality (IFM)**:

$$\lambda_{x+t} = \frac{f_x(t)}{1 - F_x(t)} \qquad t \geq 0 \tag{44}$$

cont'd

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  - $\frac{1}{b} e^{(x-m)/b}$: reflects natural death causes (increases with x and $\rightarrow \infty$ as $t \rightarrow \infty$)

cont'd

- from equation (41):

$$ {}_tp_x = e^{-\int_x^{x+t}\left(\lambda+\frac{1}{b}e^{(s-m)/b}\right)ds} = e^{-\lambda t+b(\lambda_x-\lambda)(1-e^{t/b})}; F_x(t) = 1 - {}_tp_x $$

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- we recover the PDF for the remaining lifetime r.v. $(f_x(t) = F_x'(t))$:

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- ERL under GM law of mortality is:

$$ E[\mathbf{T}_x] = \int_0^\infty e^{-\lambda t+b(\lambda_x-\lambda)(1-e^{t/b})}dt = \frac{b\Gamma(-\lambda b, b(\lambda_x-\lambda))}{e^{(m-x)\lambda+b(\lambda-\lambda_x)}} \quad (58) $$

where

$$ \Gamma(a,c) = \int_c^\infty e^{-t}t^{a-1}dt \quad (59) $$

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where v is the effective valuation rate p.a. and

$$1_{\{\mathbf{T}_x \geq t\}} = \begin{cases} 1 & \text{when} \quad \mathbf{T}_x \geq t \\ 0 & \text{when} \quad \mathbf{T}_x < t \end{cases}$$

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- the expected value of r.v. $\mathbf{a}_x$ (*immediate annuity factor*) is:

$$\bar{a}_x = E\left[\int_0^{\mathbf{T}_x} e^{-vt} dt\right] = \int_0^{\infty} e^{-vt} \; {}_t p_x \; dt = \int_0^{\infty} e^{-\left(vt + \int_0^t \lambda_{x+s} ds\right)} dt \tag{61}$$

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cont'd

- **EXAMPLE 1**: GM mortality, $\lambda = 0$, $m = 86.34$, $b = 9.5$, $v = 4\%$ and $x = 65, 75, 85 \Rightarrow \bar{a}_{65} = 12.454$, $\bar{a}_{75} = 8.718$, $\bar{a}_{85} = 5.234$

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  - the value of the annuity increases due to a longer lifespan

cont'd

- the most general (analytic) annuity factor under GM:

$$\bar{a}_x(v, T_1, T_2, m, b) := \int_{T_1}^{T_2} e^{-vt}({}_t p_x)\,dt = \int_0^{T_2 - T_1} e^{-\int_0^s (v+\lambda_{x+t})dt} ds \tag{70}$$

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- if you replace $({}_tp_x)$ or λ_{x+t}, with the relevant Gompertz-Makeham version:

$$\begin{aligned}\bar{a}_x(v, T_1, T_2, \lambda, m, b) = \frac{b}{\eta}\Gamma\left[-(\lambda+v)b, e^{\left(\frac{x-m+T_1}{b}\right)}\right] \\ - \frac{b}{\eta}\Gamma\left[-(\lambda+v)b, e^{\left(\frac{x-m+T_2}{b}\right)}\right]\end{aligned} \tag{71}$$

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$$\eta = \exp\left[(m-x)(\lambda+v) - \exp\left(\frac{x-m}{b}\right)\right] \tag{72}$$

The Problem of Retirement Income

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- we re-write the value function:

$$\max_{c_t} \int_0^{\bar{D}} e^{-\rho t} u(c_t) E[1_{\{t \le T_x\}}] dt = \max_{c_t} \int_0^{\bar{D}} e^{-\rho t} u(c_t)({}_t p_x) dt$$

  since we assume independence between optimal consumption c_t^* and the lifetime indicator function $1_{\{t \le T_x\}}$

cont'd

- the wealth (budget) constraint:

$$\dot{F}_t = v(t, F_t)F_t + \pi_0 - c_t \quad \text{with B.C.} \quad F_0 \geq 0, F_{\bar{D}} = 0 \qquad (74)$$

where π_0 is the constant income rate (pension annuity)

cont'd

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$$\dot{F}_t = v(t, F_t)F_t + \pi_0 - c_t \quad \text{with B.C.} \quad F_0 \geq 0, F_{\bar{D}} = 0 \qquad (74)$$

  where π_0 is the constant income rate (pension annuity)
- the valuation rate $v_t = v(t, F)$ is a general interest function defined by:

$$v_t = \begin{cases} v + \xi\lambda_{x+t} & F_t \geq 0, \\ \hat{v} + \lambda_{x+t}, & F_t < 0, \end{cases} \qquad (75)$$

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- **Note**: our model allows the ability to invest in actuarial notes which are instantaneous life annuities i.e. you pool your money with other people of the exact same age and the survivors gain the interest of the deceased

Euler-Lagrange Equation

- problem set-up in standard form:

$$\max_{c_t} \int_0^{\bar{D}} \phi(t, F_t, \dot{F}_t)dt \tag{76}$$

where $\phi(t, F_t, \dot{F}_t) = e^{-\rho t} u(v_t F_t - \dot{F}_t + \pi_0)\ {}_t p_x$

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- once F_t is found, we use the budget equation (74) to retrieve the *optimal consumption rate* function

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- note that v_t (defined in equation (75)) is not smooth at $F_t = 0 \Rightarrow \delta F_t$ is one-sided when $F_t = 0$

cont'd

- J reaches maximum $\Leftrightarrow \delta J \leq 0$ for both $\delta F_t > 0$ and $\delta F_t < 0$, hence:

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- combining equ's (81) and (82):

$$\frac{d}{dt} \log c_t \begin{cases} \leq k_t, & \delta F_t < 0 \\ \geq k_t, & \delta F_t > 0 \end{cases} \tag{84}$$

cont'd

- from equ's (75) , (84) and $F_t = 0$ (i.e. $c_t = \pi_0$) we get the *optimality condition*:

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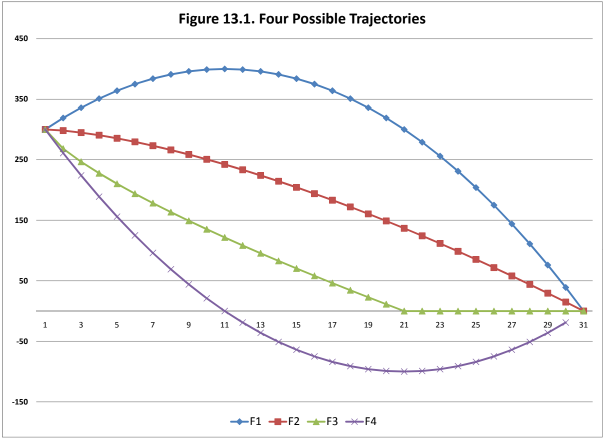
Classifying Retirement Trajectories

- **four** wealth trajectories F_t emerge from the optimization model



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   - implies a consumption rate higher than I and II
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3. **regime IV**
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   - the function F_t can take negative values
   - F_t can reach a minimum value and then increases to hit zero again at $\tau_2 \leq \bar{D}$ (the *loan depletion time (LDT)*)

Economic Cases for the Observed Trajectories

Description	**Parameters**	$\pi_0 = 0$	$\pi_0 > 0$
Relatively Patient:	$0 \leq \rho < v$	1A = [I, II]	1B = [I,II,III]
Neutral Patience:	$\rho = v < \hat{v}$	2A = [II]	2B = [II,III]
Relatively Impatient:	$v < \rho < \hat{v}$	3A = [II]	3B = [II,III]
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- **Case 4A and B**: retiree's extreme impatience, results in a very rapid and steep decline of the consumption rate

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- using the B.C.'s $F_0 = M > 0$ and $F_{\bar{D}} = 0$, we get:

$$K_1 = (M + \pi_0/v)\left(1 + \frac{e^{-k\bar{D}}}{e^{v\bar{D}} - e^{-k\bar{D}}}\right) - \left(\frac{\pi_0/v}{e^{v\bar{D}} - e^{-k\bar{D}}}\right) \tag{91}$$

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Examples: Exponential Remaining Lifetime

- **EXAMPLE 1**
  $\rho = 5\%$, $\gamma = 4$, $\lambda = 8\%$ (equivalent to a life expectancy of 12.5 yrs), $v = 4\%$, pension income $\pi_0 = \$1$, $F_0 = M = 10$, $\bar{D} = 50$ yrs, $k = 0.0225$, $K_1 = 33.069594$ and $K_2 = 1.\,9304055$

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  - F_t is **concave** and does not hit zero before $t = 50$

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Explicit Solution: Gompertz Mortality

- for Gompertz law of mortality, the survival probability:

$$({}_t p_x) = \exp\left\{ b\lambda_0 (1 - e^{t/b}) \right\} \tag{96}$$

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- from the budget equation (74), we have:

$$c_t = vF_t - \dot{F}_t + \pi_0 \tag{97}$$

$$\dot{c}_t = v\dot{F}_t - \ddot{F}_t \tag{98}$$

Explicit Solution: Gompertz Mortality

- for Gompertz law of mortality, the survival probability:

$$({}_tp_x) = \exp\left\{ b\lambda_0(1 - e^{t/b}) \right\} \tag{96}$$

where $\lambda_0 = \exp((x - m)/b)/b$ and x denotes the age at time 0

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- after rearranging equation (78):

$$\ddot{F}_t - v\dot{F}_t + k_t(vF_t - \dot{F}_t) = -k_t\pi_0 \tag{99}$$

cont'd

- after substituting equations (97) and (98) into (99):

$$k_t c_t - \dot{c}_t = 0 \tag{100}$$

cont'd

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- the *optimal solution* to equation (100) is:

$$c_t^* = c_0^* e^{\int_0^t k_s ds} \tag{101}$$

cont'd

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cont'd

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cont'd

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- the *optimal trajectory of wealth* after substituting (101) into (97) is:

$$\dot{F}_t - vF_t - \pi_0 + c_0^* e^{\left(\frac{v-\rho}{\gamma}\right)t} ({}_t p_x)^{1/\gamma} = 0 \tag{102}$$

cont'd

- after substituting equations (97) and (98) into (99):

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- after algebraic manipulations and the use of equation (71):

$$F_t = \left(F_0 + \frac{\pi}{v}\right) e^{vt} - \bar{a}_x(v-k, 0, \tau, \lambda, \hat{m}, b) c_0^* e^{vt} - \frac{\pi_0}{v} \tag{104}$$

where $\hat{m} = m + b \ln \gamma$

cont'd

- using the B.C. $F_\tau = 0$:

$$c_0^* = \frac{(F_0 + \pi_0/v)\, e^{v\tau} - \pi_0/v}{\bar{a}_x(v - k, 0, \tau, \lambda, \hat{m}, b) e^{v\tau}} \tag{105}$$

where τ is a wealth depletion time (WDT)

cont'd

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  where τ is a wealth depletion time (WDT)

- substituting equation (105) into (101) and setting $c_\tau^* = \pi_0$, we obtain an equation for τ:

$$\left(F_0 + \frac{\pi_0}{v}\right) e^{v\tau} - \frac{\pi_0}{v} = \pi_0 \bar{a}_x(v-k, 0, \tau, \lambda, \hat{m}, b)e^{(v-k)\tau} \tag{106}$$

Back to Working Years

- We expand the LCM to include wages during the *working years* (and hence the human capital).

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$$\dot{F}_t = v(t, F_t)F_t + w_t + b_t - c_t \quad \text{and} \quad F_0 = F_{\bar{D}} = 0 \tag{108}$$

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- the wage function w_t, pension income b_t and the valuation rate v are as follows:

$$w_t := \begin{cases} w_0 \exp(\rho t); & 0 \leq t \leq \bar{R} \\ 0; & t > \bar{R} \end{cases}$$

$$b_t := \begin{cases} 0; & t \leq \bar{R} \\ \pi_0; & t > \bar{R} \end{cases} \qquad v(t, F_t) = \begin{cases} v + \xi \lambda_{x+t}; & F_t \geq 0 \\ \hat{v} + \lambda_{x+t}; & F_t < 0 \end{cases}$$

cont'd

- we assume $\xi = 1$

cont'd

- we assume $\bar{\zeta} = 1$
- the optimal consumption rate is a combination of three possibilities: either c_t^* equals the wage w_t, or the pension income π_0, or is the solution of the E-L equation

$$\dot{\zeta}_t = -v(t, F_t)\zeta_t, \quad c_t^* = e^{-\frac{\rho}{\gamma}t}\zeta_t^{-\frac{1}{\gamma}} \tag{110}$$

$$\dot{F}_t = v(t, F_t)F_t + w_t + b_t - c_t^* \tag{111}$$

with $F_0 = F_{\bar{D}} = 0$

Relatively Patient Individual ($k \leq \hat{k} < g$)

- when $\bar{R} = \bar{D} \Rightarrow F_t < 0$ for $0 < t < \bar{R}$

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  Case 1: $\tau < \bar{R}$

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- first, we have $c_t^* = c_0^* \exp(\hat{k}t)$ and:

$$e^{-\hat{v}t} F_t = -\frac{e^{-(\hat{v}-g)t} - 1}{\hat{v} - g} + c_0^* \frac{e^{-(\hat{v}-\hat{k})t} - 1}{\hat{v} - \hat{k}} \qquad (112)$$

for $0 < t < \tau$

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for $0 < t < \tau$

- next, we have $c_t^* = \hat{c}_0^* \exp(kt)$ and:

$$e^{-vt} F_t = -\frac{e^{-(v-g)t} - e^{-(v-g)\tau}}{v - g} + \hat{c}_0^* \frac{e^{-(v-k)t} - e^{-(v-k)\tau}}{v - k} \tag{113}$$

for $\tau < t < \bar{R}$

cont'd

- for $\bar{R} < t < \bar{D}$

$$e^{-vt}F_t = -\pi_0 \frac{e^{-vt} - e^{-v\bar{D}}}{v} + \hat{c}_0^* \frac{e^{-(v-k)t} - e^{-(v-k)\bar{D}}}{v-k} \qquad (114)$$

- for $\bar{R} < t < \bar{D}$

$$e^{-vt}F_t = -\pi_0 \frac{e^{-vt} - e^{-v\bar{D}}}{v} + \hat{c}_0^* \frac{e^{-(v-k)t} - e^{-(v-k)\bar{D}}}{v-k} \tag{114}$$

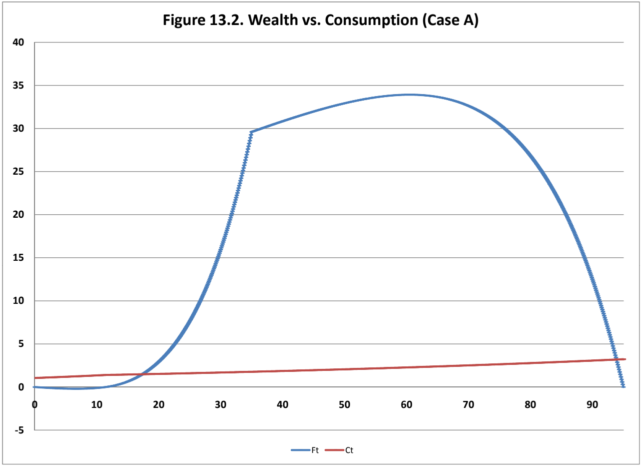
- the value of τ is the root of the function:

$$f(\tau) = \hat{c}_0^* \frac{e^{-(v-k)\tau} - e^{-(v-k)\bar{D}}}{v-k} + \frac{e^{-(v-g)\bar{R}} - e^{-(v-g)\tau}}{v-g} - \pi_0 \frac{e^{-v\bar{R}} - e^{-v\bar{D}}}{v} \tag{115}$$

where

$$\hat{c}_0^* = c_0^* e^{(\hat{k}-k)\tau}, \quad c_0^* = \frac{\hat{v}-\hat{k}}{\hat{v}-g} \frac{e^{-(\hat{v}-g)\tau} - 1}{e^{-(\hat{v}-\hat{k})\tau} - 1} \tag{116}$$

Example: $\hat{k} = 2.5\%$, $g = 3.5\%$, $v = 6\%$, $\hat{v} = 10.5\%$, $\rho = 3\%$, $\gamma = 3$, $\bar{R} = 35$, $\bar{D} = 60$ and $\pi_0 = 0.25 \Rightarrow$ it's optimal to borrow for up to $\tau = 14.85$ years



Case 2: $\tau > \bar{R}$

- first, we have $c_t^* = c_0^* \exp(\hat{k}t)$ and

$$e^{-\hat{v}t}F_t = -\frac{e^{-(\hat{v}-g)t}-1}{\hat{v}-g} + c_0^* \frac{e^{-(\hat{v}-\hat{k})t}-1}{\hat{v}-\hat{k}} \tag{117}$$

for $0 < t < \bar{R}$, and

$$e^{-\hat{v}t}F_t = -\pi_0 \frac{e^{-\hat{v}t}-e^{-\hat{v}\tau}}{\hat{v}} + c_0^* \frac{e^{-(\hat{v}-\hat{k})t}-e^{-(\hat{v}-\hat{k})\tau}}{\hat{v}-\hat{k}} \tag{118}$$

for $\bar{R} < t < \tau$

cont'd

Case 2: $\tau > \bar{R}$

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for $0 < t < \bar{R}$, and

$$e^{-\hat{v}t}F_t = -\pi_0 \frac{e^{-\hat{v}t} - e^{-\hat{v}\tau}}{\hat{v}} + c_0^* \frac{e^{-(\hat{v}-\hat{k})t} - e^{-(\hat{v}-\hat{k})\tau}}{\hat{v} - \hat{k}} \qquad (118)$$

for $\bar{R} < t < \tau$

- next, we have $c_t^* = \hat{c}^* \exp(kt)$ and

$$e^{-rt}F_t = -\pi_0 \frac{e^{-vt} - e^{-v\bar{D}}}{v} + \hat{c}^* \frac{e^{-(v-k)t} - e^{-(v-k)\bar{D}}}{v - k} \qquad (119)$$

for $\tau < t < \bar{D}$

cont'd

- the value of τ is the root of:

$$f(\tau) = \hat{c}_0^* \frac{e^{-(v-k)\tau} - e^{-(v-k)\bar{D}}}{v-k} - \pi_0 \frac{e^{-v\tau} - e^{-v\bar{D}}}{v} \tag{120}$$

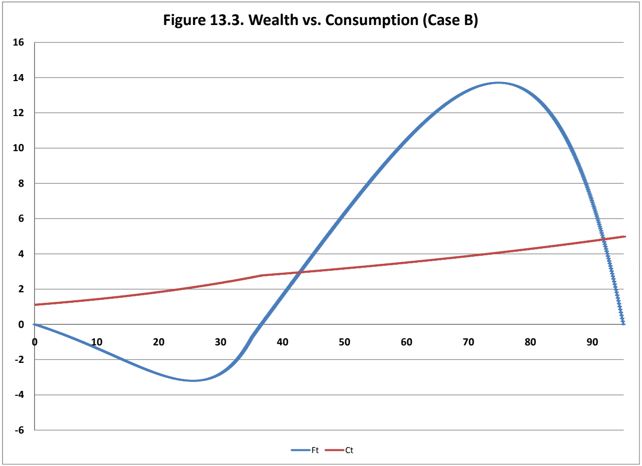
where

$$c_0^* = \frac{\hat{v} - \hat{k}}{e^{-(\hat{v}-\hat{k})\tau} - 1} \left(\frac{e^{-(\hat{v}-g)\tau} - 1}{\hat{v} - g} - P \frac{e^{-\hat{v}\bar{R}} - e^{-\hat{v}\tau}}{\hat{v}} \right)$$

and

$$\hat{c}^* = c_0^* e^{(\hat{k}-\hat{v})\tau}$$

Example: $\pi_0 = 3.25$ and fix other parameters as in Fig. 13.2$\Rightarrow$ it's optimal to borrow for up to $\tau = 39.43$ years



Relatively Impatient Individual ($k \leq g \leq \hat{k}$)

- when $\bar{R} = \bar{D}$ $\Rightarrow$one simply consumes income and maintains $F_t = 0$

Relatively Impatient Individual ($k \leq g \leq \hat{k}$)

- when $\bar{R} = \bar{D} \Rightarrow$one simply consumes income and maintains $F_t = 0$
- when $\bar{R} < \bar{D}$ and $\pi_0 = 0 \Rightarrow F_t > 0$ for $t \geq \bar{R}$

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- for $\pi_0 < \exp(g\bar{R}) \Rightarrow F_t = 0$ for $t < \tau$ and $F_t > 0$ for $t > \tau$

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  Case 1: $\tau < \bar{R}$

Relatively Impatient Individual ($k \leq g \leq \hat{k}$)

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Case 1: $\tau < \bar{R}$

- first, we have $c_t^* = \exp(gt)$ and $F_t = 0$ for $0 < t < \tau$

Relatively Impatient Individual ($k \leq g \leq \hat{k}$)

- when $\bar{R} = \bar{D}$ $\Rightarrow$one simply consumes income and maintains $F_t = 0$
- when $\bar{R} < \bar{D}$ and $\pi_0 = 0 \Rightarrow F_t > 0$ for $t \geq \bar{R}$
- for $\pi_0 < \exp(g\bar{R}) \Rightarrow F_t = 0$ for $t < \tau$ and $F_t > 0$ for $t > \tau$

Case 1: $\tau < \bar{R}$

- first, we have $c_t^* = \exp(gt)$ and $F_t = 0$ for $0 < t < \tau$
- next, we have $c_t^* = \hat{c}^* \exp(kt)$ and:

$$e^{-vt}F_t = -\frac{e^{-(v-g)t} - e^{-(v-g)\tau}}{v-g} + \hat{c}^* \frac{e^{-(v-k)t} - e^{-(v-k)\tau}}{v-k} \qquad (122)$$

for $\tau < t < \bar{R}$, and

$$e^{-vt}F_t = -\pi_0 \frac{e^{-vt} - e^{-v\bar{D}}}{v} + \hat{c}^* \frac{e^{-(v-k)t} - e^{-(v-k)\bar{D}}}{v-k} \qquad (123)$$

for $\bar{R} < t < \bar{D}$

cont'd

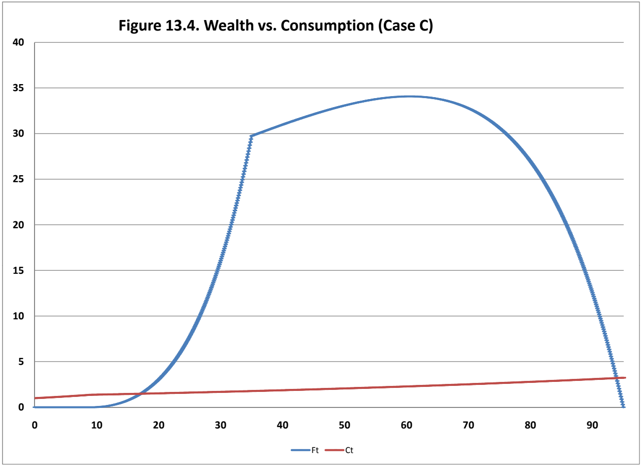
- the value of τ is the root of the function:

$$f(\tau) = \hat{c}^* \frac{e^{-(v-k)\tau} - e^{-(v-k)\bar{D}}}{v-k} + \frac{e^{-(v-g)\bar{R}} - e^{-(v-g)\tau}}{v-g} - \pi_0 \frac{e^{-v\bar{R}} - e^{-v\bar{D}}}{v} \tag{124}$$

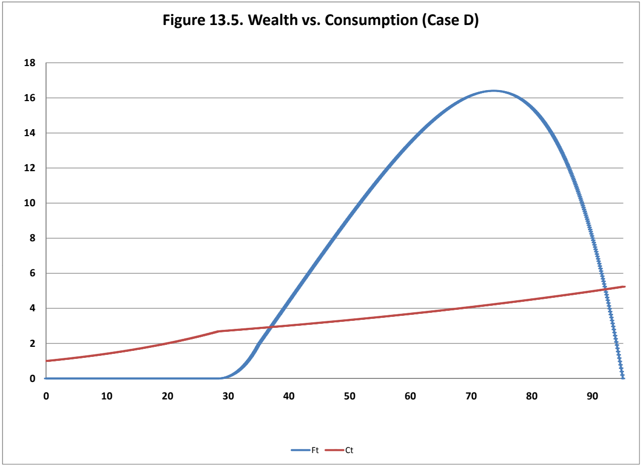
where

$$\hat{c}^* = e^{(g-k)\tau} \tag{125}$$

Example: $\pi_0 = 0.25$, $\hat{v} = 15\%$, $\hat{k} = 4\%$ and fix other parameters as in Fig. 13.2$\Rightarrow$ it's optimal to borrow for up to $\tau = 11.65$ years



Example: $\pi_0 = 3.25$, $\hat{v} = 15\%$, $\hat{k} = 4\%$ and fix other parameters as in Fig. 13.2$\Rightarrow$ it's optimal to borrow for up to $\tau = 30.19$ years



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Case 2: $\tau > \bar{R}$

- this can only occur when $\pi_0 > \exp(g\bar{R})$, i.e., the pension income is greater than the final wage just before retirement (unlikely case)

cont'd

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- this can only occur when $\pi_0 > \exp(g\bar{R})$, i.e., the pension income is greater than the final wage just before retirement (unlikely case)
- first, we have $c_t^* = \exp(gt)$ (simply consumes wage income) for $0 < t < \tau$

cont'd

Case 2: $\tau > \bar{R}$

- this can only occur when $\pi_0 > \exp(g\bar{R})$, i.e., the pension income is greater than the final wage just before retirement (unlikely case)
- first, we have $c_t^* = \exp(gt)$ (simply consumes wage income) for $0 < t < \tau$
- next, we have $c_t^* = \hat{c}^* \exp(kt)$ and:

$$e^{-vt}F_t = -\pi_0 \frac{e^{-vt} - e^{-v\bar{D}}}{v} + \hat{c}^* \frac{e^{-(v-k)t} - e^{-(v-k)\bar{D}}}{v-k} \qquad (126)$$

for $\tau < t < \bar{D}$

cont'd

Case 2: $\tau > \bar{R}$

- this can only occur when $\pi_0 > \exp(g\bar{R})$, i.e., the pension income is greater than the final wage just before retirement (unlikely case)
- first, we have $c_t^* = \exp(gt)$ (simply consumes wage income) for $0 < t < \tau$
- next, we have $c_t^* = \hat{c}^* \exp(kt)$ and:

$$e^{-vt}F_t = -\pi_0 \frac{e^{-vt} - e^{-v\bar{D}}}{v} + \hat{c}^* \frac{e^{-(v-k)t} - e^{-(v-k)\bar{D}}}{v-k} \tag{126}$$

  for $\tau < t < \bar{D}$
- the value τ is the root of the function:

$$f(\tau) = \hat{c}^* \frac{e^{-(v-k)\tau} - e^{-(v-k)\bar{D}}}{v-k} - \pi_0 \frac{e^{-v\tau} - e^{-v\bar{D}}}{v} \tag{127}$$

  where

$$\hat{c}^* = e^{(g-k)\tau} \tag{128}$$

Impatient Individual ($g < k \leq \hat{k}$)

- the optimal solution yields $F_t > 0$ (no debt)

Impatient Individual ($g < k \leq \hat{k}$)

- the optimal solution yields $F_t > 0$ (no debt)
- the solution is $c_t^* = c_0^* \exp(kt)$ and:

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for $\bar{R} \leq t \leq \bar{D}$, and

$$e^{-vt} F_t = -\frac{e^{-(v-g)t} - 1}{v-g} + c_0^* \frac{e^{-(v-k)t} - 1}{v-k} \tag{130}$$

for $0 \leq t \leq \bar{R}$, with

$$c_0^* = \frac{v-k}{e^{-(v-k)\bar{D}} - 1} \left(\frac{e^{-(v-g)\bar{R}} - 1}{v-g} - \pi_0 \frac{e^{-v\bar{R}} - e^{-v\bar{D}}}{v} \right) \tag{131}$$

Example: $k = 4\%$, $v = 15\%$ and fix other parameters as in Fig. 13.5$\Rightarrow$ F_t is positive over the entire lifecycle