A Proper Derivation of the 7 Most Important Equations for Your Retirement

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Abstract

In a recent book, Milevsky (2012) proposes seven key equations that are central to all retirement income calculations. The book itself presents these equations without deriving or proving them, and instead focuses on their history, usage and applicability. In this (brief) article I fill in the missing details by either deriving the equations themselves, or providing a detailed and specific technical reference. [Note. This is work in progress.]

1 Introduction

In a recently published book entitled: *The 7 most Important Equations for Your Retirement: The Fascinating People and Ideas Behind Planning Your Retirement Income* (June, 2012), the author proposes a number fundamental equations or calculations that all financial planners should be familiar with, in the nascent field of retirement income planning.

The book itself presents these equations (as artwork) without deriving or proving them, and instead focuses on their history, usage and applicability. The omission was partially due to the publisher's limited appetite for technical derivations. In this (brief) article I fill in the missing details by

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deriving the equations themselves as well as explaining their significance and limitations, all from first principles. [Note. This is work in progress.]

2 The 7 Equations.

In the book each one of the 7 equations is presented via – and associated with – a particular historical figure who first derived the expression or whose work is closely associated with the equation. In this article I will follow the same format.

2.1 Leonardo Fibonacci

The equation for the so-called ruin time was displayed in the book (on page #7) as:

$$t = \frac{1}{r} \ln \left[\frac{c}{c - Wr} \right], \tag{1}$$

where $r > 0$ denotes the (fixed) interest rate, $W > 0$ denotes the initial investment 'nest egg' and the variable $c > 0$ denotes the annual consumption rate. This equation (obviously) breaks-down and is not defined[1] when the denominator within the natural logarithm expression is zero or negative, i.e. when $c \leq Wr$. The intuition here is as follows. When the consumption rate c is less than the dollar (amount of) interest flowing into the investment account, the nest egg will never be depleted and $t = \infty$.

Here is a proper derivation with the relevant constraints. The present value of the retirement consumption from time $s = 0$ to any (arbitrary) time $s = t$, can be expressed as the following:

$$\text{PV} = \int_0^t ce^{-rs} ds = \frac{-c}{r} \left(e^{-rt} - e^{r0} \right) = \frac{-c}{r} \left(e^{-rt} - 1 \right). \tag{2}$$

The next step is to solve for the t at which the PV of consumption is exactly equal to W. In other words, we are looking for the t at which $W = (-c/r)(e^{-rt} - 1)$. This leads to:

$$e^{-rt} = \frac{Wr}{-c} + 1, \tag{3}$$

[1] A number of emails (to the author) expressed concern that their calculator was broken or malfunctioning because they got an error message when they tried to solve for t with a 'low' consumption rate. Mea culpa for not stressing this in the book.

which – assuming $Wr < c$ (as mentioned earlier, implying that consumption exceeds interest earnings), leads to:

$$rt = -\ln\left[\frac{Wr}{-c} + 1\right] = -\ln\left[\frac{Wr - c}{-c}\right] = \ln\left[\frac{c}{c - Wr}\right]. \tag{4}$$

Finally, divide by $-r$ one recovers the original equation (1). Q.E.D.

2.1.1 Alternative Derivation

Start with an ordinary differential equation describing the evolution of the retirement account over time. Every instant Δs, the investment account value (denoted by) w earns interest $rw\Delta t$ but experiences withdrawals $c\Delta t$. In the limit as $\Delta s \to 0$, this can be expressed as:

$$dw_s = (rw_s - c)ds, \qquad w_0 = W. \tag{5}$$

Technically speaking this ODE is only defined when $w_s \geq 0$, and one therefore must impose the condition that $s \leq t$, where $W_t = 0$. Now, the solution to this ODE is:

$$w_s = We^{rs} - c\left(\frac{e^{rs} - 1}{r}\right), \qquad s < t. \tag{6}$$

Once again, there is an implicit understanding that if-and-when $w_t = 0$, the wealth function is defined as $w_s = 0$ for all $s \geq t$. The next step is to actually solve for the time at which $W_s = 0$, based on equation (6). This leads to

$$We^{rt} = c\left(\frac{e^{rt} - 1}{r}\right), \tag{7}$$

and we must now isolate the ruin-time t. The above expression can be re-written as:

$$\left(W - \frac{c}{r}\right)e^{rt} = \frac{-c}{r}, \tag{8}$$

which then leads to:

$$e^{rt} = \frac{-c/r}{W - c/r} = \frac{c}{c - Wr}. \tag{9}$$

At this point it should be clear that since $e^{rt} > 0$, the above expression only makes sense when $c > Wr$. (There it is again.) Otherwise, $We^{rt} > (c/r)(e^{rt} - 1)$ and one can never get equality in equation (7). Finally, taking natural logarithms and dividing both side by the interest rate, one arrives at $t = (1/r)\ln[c/(c - Wr)]$ which is the original equation.

It is worth noting (again) the Fibonacci himself never wrote down this equation (or any other one, for that matter.) Logarithms had not yet been invented, let alone natural logarithms. Rather, Fibonacci's contribution was to offer the technique for computing the present value of withdrawals (or any cash-flow vector), discounted until any time in the future s. Equation #1 in the book solves for the value of s at which the PV is equal to the initial nest egg.

Another way to solve for the ruin-time – for those who refuse to use logarithms – is to employ any business calculator. Input the rate, PV (which is W) and the weekly cash-flows (which is $c/52$) and 'solve for n' which is the number of weeks until the money runs out. Divide by 52. The answer will be close enough.

2.2 Benjamin Gompertz

The equation displayed in the book (page #31) is:

$$\ln[p] = \left(1 - e^{t/b}\right) e^{\left(\frac{x-m}{b}\right)}, \tag{10}$$

where p denotes the probability of survival, m denotes the modal value and b denotes the dispersion coefficient of the Gompertz distribution. The basis for this equation is the Gompertz force of mortality denoted by:

$$\lambda_x = \frac{1}{b} e^{\left(\frac{x-m}{b}\right)}, \qquad x > 0. \tag{11}$$

Given any mortality (hazard) rate function, the survival probability can be expressed as:

$$({}_t p_x) = e^{-\int_x^{x+t} \lambda_s ds} = e^{-\int_x^{x+t} (\frac{1}{b} e^{\left(\frac{s-m}{b}\right)}) ds}, \tag{12}$$

where the (new) symbol $({}_t p_x)$ denotes the probability an x-year old will survive for t more years. After some rather tedious calculus (change of variables within the integral) this leads to equation (10). For a more detailed step-by-step derivation of the calculus, please see the discussion in Milevsky (2006) from page #38 to page #49, and in particular equation (3.25) in that book. Another source is Charupat, Huang and Milevsky (2012), page #286 to #287.

2.3 Edmond Halley

The equation displayed in the book (page #53) is:

$$a_x = \sum_{i=1}^{\infty} \frac{{}_i p_x}{(1+R)^i}, \tag{13}$$

where $({}_i p_x)$ denotes the survival probability and R is the periodic valuation rate. The period in question (i) can be a year, month, week or instant. Either way the mathematics is the same, as long as the payment is $1 per period. The justification for this formula comes from the discounting of the *expected payment* at the end of the period. For more details see Milevsky (2006) page #114. Once again, Halley himself never wrote down this expression, but rather described it in words. There is a strong connection between the annuity factor and expectation of life, in that they are equal when the interest rate is zero. See the article by James Ciecka (2011) on the history behind the proper calculation of life expectancy. Indeed, one could argue it deserves to be the 8th most important equation for retirement income planning. The heroes there would be the Huygens brothers and their discovery in 1669.

2.4 Irving Fisher

The equation displayed in the book (page #77) is:

$$\ln[c_{x+1}] - \ln[c_x] = \frac{r - \rho + \ln[p_x]}{\gamma}, \tag{14}$$

where c_x is the consumption at age x, and r denotes the interest rate, ρ denotes the subjective discount rate, p_x denotes the survival probability and γ denotes the risk aversion coefficient. This equation can be traced to Yaari's (1965) interpretation and representation of Fisher equation. In fact, the equation should probably be called the Fisher-Yaari equation, perhaps with the name Ramsey (1928) attached for good measure. A proper derivation is in the original Yaari (1965) paper, or in the more recent book by Charupat, Huang and Milevsky (2012), page #291 to #293.

2.5 Paul Samuelson

The equation displayed in the book (page #101) is:

$$\Psi = \frac{1}{\gamma}(HC + FC)\left(\frac{\mu - r}{\sigma^2}\right), \tag{15}$$

where the left-hand side denotes the dollar value allocated to (risky) stocks, the symbol HC denotes the value of human capital, FC denotes the value of financial capital, μ denotes the expected return from (risky) stocks, r denotes the risk-free rate and σ^2 denotes the volatility of (risky) stocks. A variant of this equation can be found in the original article by Samuelson (1969), as well as the book by Merton (1990). Indeed, this equation could have (should have?) been called the Samuelson-Merton equation. For a proper and careful derivation, please see page #323 in the book by Charupat, Huang and Milevsky (2012) and in particular equation (14.31) in Section 14.6. Note the implict assumption that HC is traded and hence can be treated in the same category as FC. This, of course, is a questionable and debatable assumption, especially after the 1863 Emancipation Proclamation.

2.6 Solomon S. Huebner

The equation displayed in the book (page #125) was:

$$A_x = \sum_{i=1}^{\infty} \frac{({}_ip_x)(q_{x+i})}{(1+R)^{i+1}}, \tag{16}$$

where $({}_ip_x)$ denotes the survival probability, (q_{x+i}) denotes the one-period mortality rate and R denotes the valuation rate. The justification for this equation comes from the discounting of cash-flows for mortality and interest. See Milevsky (2006), page #144 for a proper derivation under general mortality, as well as under the specific case of Gompertz-Makeham mortality. As mentioned in the book (and the appendix) this equation itself likely preceeded Huebner by a century. If pressed, the British actuary Richard Price (1723-1791) would be the second (and perhaps) first name associated with this equation.

2.7 Andrei N. Kolmogorov

The equation displayed in the book (page #151) was:

$$P\lambda_t = \frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial w} + \frac{1}{2}\sigma^2 w^2 \frac{\partial^2 P}{\partial w^2}, \tag{17}$$

where P denotes the lifetime ruin probability (LRP), μ denotes the expected return from the risky asset, σ denotes the volatility and w denotes the current level of investable wealth scaled by the spending rate. The idea here is to assume that W_t obeys the following diffusion:

$$dW_t = (\mu W_t - 1)dt + \sigma W_t dB_t, \tag{18}$$

where B_t is a Brownian motion. The probability that the ruin time of the diffusion (denoted by $\mathbf{R}_w$) is smaller than the remaining lifetime random variable (denoted by $\mathbf{T}_x$), leads to the PDE in equation (17). This can be derived using martingale arguments, namely, that the conditional expectation process is a maringale, and hence the drift term must be zero. For more details please see Milevsky (2006) page 209 to 211, as well as Huang, Milevsky and Wang (2004), page 422 to 425. In particular, see the discussion leading up to equation (21).

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