Strategic Financial Planning over the Lifecycle
Chapter #2: Mathematical Preliminaries

Narat Charupat, Huaxiong Huang and Moshe A. Milevsky

Ch. #2: Lecture Notes
The only mathematics you need for this course

- The **future** value of a $1 bullet payment to be received after $N$-periods, assuming a valuation rate of $v\%$ per period, is:

  \[ FV(v, N) \]
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The present value of a $1 bullet payment to be received after \(N\)-periods, assuming a valuation rate of \(v\%\) per period, is:

\[ PV(v, N) \]
The **future** value of a $1 bullet payment to be received after $N$-periods, assuming a valuation rate of $v\%$ per period, is:

$$FV(v, N)$$

The **present** value of a $1$ bullet payment to be received after $N$-periods, assuming a valuation rate of $v\%$ per period, is:

$$PV(v, N)$$

The **future** value of a $1$ annuity payment, growing at $g\%$ per period for $N$-periods, assuming a valuation rate of $v\%$ per period:

$$FVA(g, v, N)$$
The only mathematics you need for this course

- The **future** value of a $1 bullet payment to be received after \(N\)-periods, assuming a valuation rate of \(v\%\) per period, is:

\[
FV(v, N)
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- The **present** value of a $1 annuity payment, growing at \(g\%\) per period for \(N\)-periods, assuming a valuation rate of \(v\%\) per period:

\[
PVA(g, v, N)
\]
To obtain these values, you can (blindly) plug numbers into your business calculator, MS Excel or you can use the following analytic expressions.

The bullet-payment present and future value are trivially:

$$PV(v, N) = (1 + v)^N$$
$$FV(v, N) = (1 + v)^N$$

The present value annuity “factor” is obtained as follows:

$$PVA(g, v, N) = \sum_{j=1}^{N} \frac{(1 + g)^j}{(1 + v)^j}$$

The future value annuity “factor” is then:

$$FVA(g, v, N) = PVA(g, v, N) \cdot FV(v, N) = (1 + v)^N \cdot \left( \frac{v g}{1 + g} \right)$$

Note that when $$v = g$$ the present value is (trivially) $$PVA = N$$ and $$FVA = N(1 + v)^N$$.
To obtain these values, you can (blindly) plug numbers into your business calculator, MS Excel or you can use the following analytic expressions.

The bullet-payment present and future value are trivially:

\[ PV(v, N) = (1 + v)^{-N}, \quad FV(v, N) = (1 + v)^N \]
To obtain these values, you can (blindly) plug numbers into your business calculator, MS Excel or you can use the following analytic expressions.

The bullet-payment **present** and **future** value are trivially:

$$PV(v, N) = (1 + v)^{-N}, \quad FV(v, N) = (1 + v)^N$$

The **present** value annuity "factor" is obtained as follows:

$$PVA(g, v, N) = \sum_{j=1}^{N} (1 + g)^j (1 + v)^{-j} = \frac{1 - (1 + g)^N (1 + v)^{-N}}{(v - g)/(1 + g)}$$

Note that when $v = g$ the present value is (trivially) $PVA = N$ and $FVA = N (1 + v)^N$. 

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**Time Value of Money (TVM) Analytics**

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**Strategic FP over L**

**Ch. #2: Lecture Notes**
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  PVA(g, v, N) = \sum_{j=1}^{N} (1 + g)^j (1 + v)^{-j} = \frac{1 - (1 + g)^N (1 + v)^{-N}}{(v - g) / (1 + g)}
  \]

- The future value annuity "factor" is then:
  \[
  FVA(g, v, N) = PVA(g, v, N) \times FV(v, N) = \frac{(1 + v)^N - (1 + g)^N}{(v - g) / (1 + g)}
  \]
To obtain these values, you can (blindly) plug numbers into your business calculator, MS Excel or you can use the following analytic expressions.

The bullet-payment present and future value are trivially:

\[ \text{PV}(v, N) = (1 + v)^{-N}, \quad \text{FV}(v, N) = (1 + v)^N \]

The present value annuity "factor" is obtained as follows:

\[ \text{PVA}(g, v, N) = \sum_{j=1}^{N} (1 + g)^j (1 + v)^{-j} = \frac{1 - (1 + g)^N (1 + v)^{-N}}{(v - g) / (1 + g)} \]

The future value annuity "factor" is then:

\[ \text{FVA}(g, v, N) = \text{PVA}(g, v, N) \times \text{FV}(v, N) = \frac{(1 + v)^N - (1 + g)^N}{(v - g) / (1 + g)} \]

Note that when \( v = g \) the present value is (trivially) \( \text{PVA} := N \) and \( \text{FVA} := N(1 + v)^N \).
Question #1: Assume that you are earning a salary of $5,000 per month, paid at the end of the month. This grows by $g = 0.03/12 = 0.0025$ per month (due to inflation). What is the present value of your lifetime salary assuming a monthly valuation rate of $\nu = 0.06/12 = 0.005$ and that you work for another $N = 30$ years?
Question #1: Assume that you are earning a salary of $5,000 per month, paid at the end of the month. This grows by $g = \frac{0.03}{12} = 0.0025$ per month (due to inflation). What is the present value of your lifetime salary assuming a monthly valuation rate of $\nu = \frac{0.06}{12} = 0.005$ and that you work for another $N = 30$ years?

Answer #1: The annuity factor is:

$$PVA(0.0025, 0.005, 360) = \sum_{j=1}^{360} (1 + 0.0025)^j (1 + 0.005)^{-j} = \frac{1 - (1 + 0.0025)^{360}(1 + 0.005)^{-360}}{(0.0050 - 0.0025)/(1 + 0.0025)} = \$237.4165$$

So, the present value of your lifetime salary is: $237.4165 \times 5000 = \$1,187,083$.

Note that on the day you retire (at the end of month $N = 360$) you salary will be $5000 \times (1 + \frac{0.03}{12})^{360} = \$12,284$. 

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**Examples: Part I**

- **Question #1**: Assume that you are earning a salary of $5,000 per month, paid at the end of the month. This grows by $g = 0.03/12 = 0.0025$ per month (due to inflation). What is the present value of your lifetime salary assuming a monthly valuation rate of $\nu = 0.06/12 = 0.005$ and that you work for another $N = 30$ years?

- **Answer #1**: The annuity factor is: 

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$$\frac{1 - (1 + 0.0025)^{360} (1 + 0.0050)^{-360}}{(0.0050 - 0.0025)/(1 + 0.0025)} = \$237.4165$$

So, the present value of your lifetime salary is: $237.4165 \times 5000 = \$1,187,083$. 
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$$\frac{1 - (1 + 0.0025)^{360}(1 + 0.0050)^{-360}}{(0.0050 - 0.0025)/(1 + 0.0025)} = \$237.4165$$

So, the present value of your lifetime salary is: $237.4165 \times 5000 = \$1,187,083$.

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Strategic FP over L  
Ch. #2: Lecture Notes
**Question #2**: In contrast to the previous question, assume that you are paid $60,000 plus $g = 3\%$ inflation at year-end. What is the present value at $v = 6\%$ over $N = 30$ years?
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Answer #2: In this case the annuity factor is:

$$PVA(0.03, 0.06, 30)$$

So the present value of your time salary is: $19.82369 \times 60,000 = \$1,189,421$. Notice that by compounding less frequently, we increased the present value (slightly).
Question #2: In contrast to the previous question, assume that you are paid $60,000 plus $g = 3\%$ inflation at year-end. What is the present value at $\nu = 6\%$ over $N = 30$ years?

Answer #2: In this case the annuity factor is:

$$\text{PVA}(0.03, 0.06, 30)$$

So the present value of your time salary is: $19.82369 \times 60,000 = \$1,189,421$. Notice that by compounding less frequently, we increased the present value (slightly).

Note that with zero salary growth (i.e. no inflation adjustment) the present value of salary is (only):

$$\text{PVA}(0, 0.06, 30) \times 60,000 = \$825,890.$$
Question #3: You won the $1,000,000 lottery which you are saving in a bank account earning $v = 5\%$ per year. You plan to withdraw cash-flows of $70,000$ per year, at year-end. How many years will the money last?

Answer #3: You must locate the value of $N$ that solves the following equation:

$$PVA(0, 0.05, N) = 1,000,000 - 70,000$$

Using solver (in Excel) or any other numerical iteration technique, the solution to the above equation is:

$$N = 25.68$$

So, the funds will last until the 25th year. At the end of the 26th year you will not be able to withdraw the entire $70,000.
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\[
PVA(0, 0.05, N) = \frac{1,000,000}{70,000}
\]
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PVA(0, 0.05, N) = \frac{1,000,000}{70,000}
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Using solver (in Excel) or any other numerical iteration technique, the solution to the above equation is: \( N = 25.68 \) years. So, the funds will last until the 25th year. At the end of the 26th year you will not be able to withdraw the entire $70,000.
Question #4: You just won the $1,000,000 lottery which you are saving in a bank account earning $v = 5\%$ per year. You plan to withdraw cash-flows $C_j; j = 1..25$ at the end of the year for exactly 25 years, where $C_j = C_0(1 + 0.04)^j$. What is $C_0$?
Example #4: You just won the $1,000,000 lottery which you are saving in a bank account earning $v = 5\%$ per year. You plan to withdraw cash-flows $C_j; j = 1..25$ at the end of the year for exactly 25 years, where $C_j = C_0 (1 + 0.04)^j$. What is $C_0$?

Answer #4: You must locate the value of $C_0$ that solves the following equation:

$$C_0 = \frac{1,000,000}{PVA(0.04, 0.05, 25)} = $45,191.21$$
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Answer #4: You must locate the value of $C_0$ that solves the following equation:

$$C_0 = \frac{1,000,000}{PVA(0.04, 0.05, 25)} = \$45,191.21$$

Note that the first year’s cash-flow (withdrawal) will be $C_1 = 45191.21(1 + 0.04) = \$46,998.86$
Examples: Part IV

Question #4: You just won the $1,000,000 lottery which you are saving in a bank account earning $v = 5\%$ per year. You plan to withdraw cash-flows $C_j; j = 1..25$ at the end of the year for exactly 25 years, where $C_j = C_0(1 + 0.04)^j$. What is $C_0$?

Answer #4: You must locate the value of $C_0$ that solves the following equation:

$$C_0 = \frac{1,000,000}{PVA(0.04, 0.05, 25)} = $45,191.21$$

Note that the first year’s cash-flow (withdrawal) will be $C_1 = 45191.21(1 + 0.04) = $46,998.86

The second year’s cash-flow (withdrawal) will be $C_2 = 45191.21(1 + 0.04)^2 = $48,878.81
The Spreadsheet View

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Final Remarks

As you "play" with these expressions, remember the following.

- The cash-flows are (always) assumed to occur at the end of the period. If you are dealing with a situation in which cash-flows occur at different time frames, don’t panic; modify the formula.
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- The cash-flows are (always) assumed to occur at the end of the period. If you are dealing with a situation in which cash-flows occur at different time frames, don’t panic; modify the formula.

- There are many techniques and routines that can be used to "solve for" the growth rate \( g \), valuation rate \( v \), number of periods \( N \), or the \textbf{PVA} and \textbf{FVA} factors. Make sure you understand what you are doing.

Don’t let the mathematics distract you from the (bigger) financial picture.
Final Remarks

As you "play" with these expressions, remember the following.

- The cash-flows are (always) assumed to occur at the end of the period. If you are dealing with a situation in which cash-flows occur at different time frames, don’t panic; modify the formula.

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